Seat No. : _______________

MN-148

March-2019

M.Sc., Sem.-IV

509 : Mathematics
(Number Theory)
(New)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following questions : **14**

(1) Find all solutions in the positive integers of $54x + 21y = 906$.

(2) Let x and y be real numbers. Prove the following :

(i) For any integer n, $[x + n] = [x] + n$.

(ii) $[x + y] \geq [x] + [y]$.

OR

(1) If $n = p_1^{k_1} p_2^{k_2} .... p_r^{k_r}$ is the prime factorization of n > 1, prove that

$\tau(n) = (k_1 + 1)(k_2 + 1) .... (k_r + 1)$.

(2) Prove that if $2^k - 1$ is prime (k > 1), then $n = 2^{k-1}(2^k - 1)$ is perfect and every even perfect number is of this form.

(B) Attempt any **four**. **4**

(1) Write down the prime factorization of 11 !.

(2) Find the highest power of 5 dividing 2000 !

(3) What is value of $\sum_{d|40} \phi(d)$?

(4) Calculate σ (957).

(5) For any integer $n \geq 3$, what is the value of $\sum_{k=1}^{n} \mu(k!)$?

(6) True/False: A perfect square cannot be a perfect number.

2. (A) Answer the following questions : **14**

(1) State Chinese remainder theorem. (Do not prove).

Solve : $x \equiv 5 \pmod{11}$, $x \equiv 14 \pmod{29}$, $x \equiv 5 \pmod{31}$.

(2) State and prove Wilson's theorem. Does the converse true ? Justify your answer.

OR

(1) Find all the primitive roots of 17.

(2) Using the theory of indices, solve $9x^8 \equiv 8 \pmod{17}$.

(B) Attempt any **four** : **4**

(1) Is the congruence $28\,x \equiv 15 \pmod{35}$ solvable ?

(2) True/False : The units digit of 3^{100} is 3.

(3) State Euler's theorem. (Do not prove.)

(4) What is the remainder when 15! is divided by 17 ?

(5) What is the order of 3 modulo 13 ?

(6) Is 3 a primitive root of 19 ?

3. (A) Answer the following questions : **14**

(1) Solve the quadratic congruence $5x^2 + 6x + 1 \equiv 0 \pmod{23}$.

(2) State and prove Euler's criterion.

OR

(1) If p is an odd prime, prove that $\sum_{a=1}^{p-1} (a/p) = 0$.

(2) Solve the quadratic congruence $x^2 \equiv 23 \pmod{7^3}$.

(B) Attempt any **three** : **3**

(1) Is 3 a quadratic residue of 23 ?

(2) True/False : 8 is a quadratic residue of 13.

(3) Find the value of the Legendre symbol (–23/59).

(4) State quadratic reciprocity law. (Do not prove.)

(5) Evaluate the Jacobi symbol (21/221).

4. (A) Answer the following questions : **14**

(1) Solve the Diophantine equation $364x + 227y = 1$ by means of continued fraction.

(2) Determine the infinite continued fraction representation of $\frac{5+\sqrt{37}}{4}$.

OR

(1) Find the fundamental solution of $x^2 - 41y^2 = 1$.

(2) Evaluate $\left[1;2,\overline{3,1}\right]$.

(B) Attempt any **three**. **3**

(1) Express 71/55 as a simple continued fraction.

(2) True/False : Even-numbered convergents form a strictly decreasing sequence.

(3) Evaluate $\left[\overline{2,3}\right]$.

(4) If $x_1 = 6$ and $y_1 = 1$ forms the fundamental solution of the equation $x^2 - 35y^2 = 1$, then find its second positive solution.

(5) Find all primitive Pythagorean triples x, y, z in which $x = 30$.

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MN-148

March-2019

M.Sc., Sem.-IV

509 : Mathematics
(Mathematical Methods)
(Old)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following questions : **14**

(1) Solve : $x(x-1)\,y'' + (3x-1)\,y' + y = 0$.

(2) Show that $\frac{d}{dx}[x^{v} J_{v}(x)] = x^{v} J_{v-1}(x)$.

OR

(1) Find the Eigenvalues and Eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y'(1) = 0.$$

(2) Using the indicated substitutions, reduce the following equation to Bessel's differential equation and find a general solution in terms of Bessel functions.

$$x^2 y'' + xy' + \left(4x^4 - \frac{1}{4}\right) y = 0 \quad (x^2 = z).$$

(B) Attempt any **four**. **4**

(1) Write down the indicial equation of $4xy'' + 2y' + y = 0$.

(2) What is the radius of convergence of $\sum_{n=0}^{\infty} (n+1)\, n x^{n}$?

(3) True/False : $J_{v-1}(x) - J_{v+1}(x) = 2J'_{v}(x)$.

(4) What is the value of $\Gamma(1/2)$?

(5) What is the value of $J_{1/2}(\pi/2)$?

(6) True/False; $J_0(0) = 1$.

2. (A) Answer the following : **14**

(1) Using the Laplace transform, solve the following initial value problem.

$y'' - 5y' + 6y = 4e^t$ if $0 < t < 2$ and 0 if $t > 2$;, $y(0) = 1$, $y'(0) = -2$.

(2) Find the inverse Laplace transform of $\dfrac{e^{-2\pi s}}{s^2 + 2s + 2}$.

OR

(1) Using the second shifting theorem, find the Laplace transform of $t^2u(t-1)$ and $(t-1)\,u(t-1)$.

(2) Find the inverse Laplace transform of $\dfrac{s}{(s^2 - 9)^2}$.

(B) Attempt any **four** : **4**

(1) What is the Laplace transform of $\cos \omega t$?

(2) What is the Laplace transform of e^{2t} ?

(3) Find the Laplace transform of $u(t-5)$.

(4) True/False : $\mathscr{L}(f'') = s^2 \mathscr{L}(f) - sf(0) - f'(0)$.

(5) State first shifting theorem.

(6) State second shifting theorem.

3. (A) Answer the following : **14**

(1) Find the Fourier series of the function

$$f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$$

Using the Parseval's identity prove that $1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots = \dfrac{\pi^2}{8}$.

(2) Find the Fourier cosine integral of

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

OR

(1) Show that

$$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

(2) Find the Fourier transform of xe^{-x^2}.

(B) Attempt any **three** : **3**

(1) True/False : The smallest positive period p of $\cos 2x$ is π.

(2) True/False : The function $f(x) = |x^3|$ is even.

(3) Define Fourier cosine transform.

(4) True/False : $\mathscr{F}_s\{f'(x)\} = -w\mathscr{F}_c\{f(x)\}$.

(5) What is the value of the integral $\int_{-\pi}^{\pi} \cos^4 x\, dx$?

4. (A) Answer the following : **14**

(1) Find the inverse Z-transform of $\dfrac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$, when

(i) $\frac{1}{3} < |z| < \frac{1}{2}$

(ii) $\frac{1}{2} < |z|$

(2) Find the Hankel transform of

$$f(x) = \begin{cases} a^2 - x^2, & 0 < x < a \quad n = 0 \\ 0, & x > a \qquad n = 0. \end{cases}$$

OR

(1) Find the Z-transform of $c^k \cos ak$, $k \geq 0$.

(2) Prove that

$$H_n\left\{\frac{d^2f}{dx^2}\right\} = \frac{s^2}{4}\left[\frac{n+1}{n-1} H_{n-2}(s) - 2\frac{n^2-3}{n^2-1} H_n(s) + \frac{n-1}{n+1} H_{n+2}(s)\right].$$

(B) Attempt any **three** : **3**

(1) State initial value theorem.

(2) What is the Z-transform of the sequence $\left\{\frac{1}{2^k}\right\}, -4 \le k \le 4$?

(3) What is the order of the difference equation $6y_{k+2} - y_{k+1} - y_k = 0$?

(4) Define Hankel transform.

(5) What is the value of the integral $\int_0^\infty e^{-ax} J_0(sx)\, dx$?