No. : ____________

MM-155

March-2019

M.Sc., Sem.-IV

508 : Mathematics

(Algebra-II)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following questions : **14**

(1) Let R be a commutative ring with unity and let A be an ideal of R. Then prove that R/A is a field if and only if A is maximal.

(2) Prove that $M_{\frac{1}{2}} = \{f \in C[0, 1] / f\left(\frac{1}{2}\right) = 0\}$ is a maximal ideal in C[0,1].

OR

(1) Prove that a finite integral domain is a field.

(2) Let x and y belong to a commutative ring R with prime characteristic p.

(a) Show that $(x + y)^p = x^p + y^p$.

(b) Show that, for all positive integers n, $(x + y)^{p^n} = x^{p^n} + y^{p^n}$.

(B) Attempt any **four** : **4**

(1) Find all the units of the ring of polynomials $\mathbb{Z}_p[x]$. (pis prime here)

(2) Let R = C[0, 1]. Show that the ring R has zero-divisors.

(3) Give an example of an infinite ring with finite characteristic.

(4) Give an example of a ring (not a field) which has infinitely many units.

(5) Define the Boolean ring. Is it commutative ? Justify.

(6) Give an example of a prime ideal that is not maximal.

2. (A) Answer the following questions : **14**

(1) State and prove the mod p irreducibility test.

(2) Let F be a field and let $I = \{f(x) \in F[x] / f(a) = 0 \text{ for all } a \in F\}$.

Prove that I is an ideal in F[x]. Prove that I is infinite when F is finite, and I = {0} when F is infinite.

OR

(1) Let F be a field and let $p(x) \in F[x]$. Then prove that $< p(x) >$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F.

(2) Let $f(x) = 21x^3 - 3x^2 + 2x + 9$ be a polynomial in $\mathbb{Q}[x]$. Is this reducible over $\mathbb{Q}$? Justify.

(B) Attempt any **four** : **4**

(1) Determine all ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$.

(2) Can we have a non-zero polynomial $f(x) \in \mathbb{R}[x]$ such that $f(n) = 0$ for each $n \in N$? Justify.

(3) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0 \in \mathbb{Z}[x]$ and $a_n \neq 0$. If r and s are relatively prime integers such that $f\left(\frac{r}{s}\right) = 0$ then prove that $r|a_0$ and $s|a_n$.

(4) Find infinitely many polynomials $f(x)$ in $\mathbb{Z}_3[x]$ such that $f(a) = 0$ for all $a \in \mathbb{Z}_3$.

(5) Prove or disprove : $\mathbb{R}$ is ring isomorphic to $\mathbb{C}$.

(6) Let $f(x) \in \mathbb{Q}[x]$ such that $f(\pi) = 0$. What can be said about the polynomial $f(x)$? Explain.

3. (A) Answer the following questions : **14**

(1) Show that for each prime p and each positive integer n, there is a unique field of order p^n.

(2) Prove that $x^6 - 2$ has a zero in $Q(\sqrt[6]{2})$ but it does not split in $Q(\sqrt[6]{2})$. Find the splitting field of $x^6 - 2$ over $\mathbb{Q}$.

OR

(1) If E is a finite extension of F, prove that E is an algebraic extension of F. What can you say about the converse ?

(2) Define the splitting field. Find the splitting field of $x^4 - 6x^2 - 7$ over $\mathbb{Q}$.

(B) Attempt any **three** : **3**

(1) What is the dimension of GF(128) over GF(16) ?

(2) Describe the elements of $Q(\pi)$.

(3) Let $F = \mathbb{Q}(\pi^3)$. Find a basis for $F(\pi)$ over F.

(4) Show that ab is constructible if a and b are constructible.

(5) If F is a field of order 64, determine all the subfields of F.

4. (A) Answer the following questions : **14**

(1) Show that there is a quintic polynomial over $\mathbb{Q}$ that cannot be solved by radicals.

(2) Let E be an extension field of $\mathbb{Q}$. Show that any automorphism of E acts as the identity on $\mathbb{Q}$.

OR

(1) Describe the group Gal $(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q})$. (Here, ω is a primitive cube root of unity.)

(2) Define : (i) the Galois group of E over F(ii) Fixed field E_H of H (iii) the solvable group.

(B) Attempt any **three** : **3**

(1) Find the dimension of $\mathbb{Q}(\sqrt[3]{2}, \omega)$ over $\mathbb{Q}$ (Here, ω is a primitive cube root of unity.)

(2) Is it possible to find a field F which has exactly 6 sub-fields ? Justify.

(3) Let $f(x) \in F[x]$. When we say that $f(x)$ is solvable by radicals over the field F ?

(4) Prove or disprove : 15 degree angle is constructible by straightedge, compass and a unit length.

(5) True or false: The real number $\sqrt[5]{3}$ is constructible.

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MM-155

March-2019

M.Sc., Sem.-IV

508 : Mathematics

(Fourier Analysis) (Old)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following : **14**

(1) If $\in L^1(\mathbb{T})$ then show that $\int_0^{2\pi} f = \int_a^{a+2\pi} f$ for any $a \in \mathbb{R}$. Also show that for essentially bounded functions, $\| T_a(f) \|_{L\infty} = \| f \|_{L^\infty}$.

(2) State and prove Riemann Lebesgue Lemma.

OR

(1) Evaluate the Fourier series of $f : [-\pi, \pi)$, $f(x) = x$.

(2) State and prove Uniqueness theorem for 2π periodic continuous functions.

(B) Attempt any four : **4**

(1) Let $f(x) = e^{-5x} + e^{5x}$ then $\hat{f}(5) =$

(a) 0 (b) –1

(c) 10 (d) 1

(2) The Fourier transformation map $T : L^1 \to C_0(\mathbb{Z})$, $T(f) = \hat{f}$ is

(a) One to one (b) onto

(c) (a) and (b) both (d) None of the above

(3) Let $a_n = \frac{(-1)^n}{n^2}$ and $b_n = \frac{1}{n^2}$ then

(a) $a_n = O(b_n)$ (b) $b_n = o(a_n)$

(c) $a_n = o(b_n)$ (d) None of the above

(4) Let $a_n = \frac{\sin n}{n}$ then

(a) $a_n = O\left(\frac{1}{n}\right)$ (b) $a_n = O\left(\frac{1}{n^2}\right)$

(c) $a_n = o\left(\frac{1}{n^2}\right)$ (d) None of the above

(5) Let f, g $\in L^1(\mathbb{T})$ and $f(x) = -g(x)$ almost everywhere in $\mathbb{T}$ then

(a) $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$

(b) $\hat{f}(n) = -\hat{g}(n)$ for all $n \in Z$

(c) $\hat{f}(n) = -\hat{g}(n)$ only for all $n \in N$

(d) None of the above

(6) Let $f \in L^1(\mathbb{T})$ and $f(-x) = -f(x)$ for all $x \in [-\pi, \pi]$ then

(a) $\hat{f}(0) = 0$ (b) $\hat{f}(-n) = -\hat{f}(n)$ for all $n \in Z$

(c) $\hat{f}(n) = 0$ for all $n \in Z$ (d) None of the above

2. (A) Answer the following : **14**

(1) Let f, $g \in L^1$ and $a \in \mathbb{R}$. Then show that $T_a f * g = f * T_a g = T_a(f * g)$.

(2) Define a function of bounded variation. Let $f \in L^1$ and g is of bounded variation then show that f * g is of bounded variation and $V(f * g) \leq \| f \|_1 V(g)$.

OR

(1) If $f \in L^1$ and $g \in L^p$, $1 \leq p \leq \infty$. Then $f * g \in L^1$ and $\|f * g\| \leq \| f \|_1 \| g \|_p$.

(2) Define algebra homomorphism. Let $\gamma_N : L^1 \to C$ by $\gamma_N(f) = \hat{f}(N)$. Show that for any non trivial continuous complex algebra homomorphism of L^1 there exist a unique integer N such that $\gamma = \gamma_N$.

(B) Attempt any **four** : **4**

(1) Which of the following is true for $(L^1(\mathbb{T}), +, *)$, where * is a convolution operation.

Let $\{K_n\}_{n=1}^{\infty}$ be any approximate identity in L^1 and $f \in C$ then $\lim_{n\to\infty} \|K_n * f - f\|_\infty =$.

(a) 1 (b) 0

(c) -1 (d) None of the above

(2) Let γ be a complex algebra homomorphism on L^1 and

$$f(x) = \begin{cases} x & \text{if } x^2 \in \mathbb{Q} \cap [-\pi,\pi] \\ 0 & \text{if } x^2 \in \mathbb{Q}^C \cap [-\pi,\pi] \end{cases} \text{ then.}$$

(a) $\gamma(f) = 0$ (b) $\gamma(f^2) \neq 0$

(c) $\gamma(f - f^2) \neq 0$ (d) None of the above

(3) Which of the following is false for $(L^1(\mathbb{T}), +, *)$, where * is a convolution operation.

(a) $(L^1(\mathbb{T}), +, *)$ has an approximate identity.

(b) $(L^1(\mathbb{T}), +, *)$ has a zero divisor.

(c) $(L^1(\mathbb{T}), +, *)$ is a vector space.

(d) None of the above

(4) Define approximate identity in $(L^1\mathbb{T}), +, *)$.

(5) Give an example of approximate identity.

(6) Give an example of Complex algebra homomorphism of L^1.

3. (A) Answer the following : **14**

(1) Define Cesaro summable series. If $c_n = o(\frac{1}{n})$ and $\sigma_N \to l$, then show that $S_N \to l$.

(2) Show that summable series is Cesaro summable.

OR

(1) Let F_n be Fejer kernels. Show that F_N is an approximate identity.

(2) Let D_n be Dirichlet kernels. Show that $\| D_N \|_1 = \frac{4}{\pi^2} \log N + O(1)$.

(B) Attempt any **three** : **3**

(1) True/False : Let $a_n = (-1)^n$. Then a_n is Cesaro summable.

(2) Give a relation between Cesaro summability and summability.

(3) Give an example of sequence which is Cesaro summable but not summable.

(4) Write any one property of Dirichlet kernel.

(5) Write any one property of Fejer Kernel.

4. (A) Answer the following : **14**

(a) Show that Trigonometric polynomials are dense in L^p, $1 \leq p < \infty$.

(b) State and prove Localization principle.

OR

(a) Show that Fourier series converges uniformly if and only if $a_n = o(\frac{1}{n})$.

(b) State and prove Dini's test.

(B) Attempt any **three** : **3**

(1) Define a Sequence of bounded variation.

(2) Give an example of sequence of bounded variation.

(3) State Jordan's Theorem.

(4) State Taibleson's theorem.

(5) State Uniform boundedness principle.
