Seat No. : _______________

ML-130

March-2019

M.Sc., Sem.-IV

507 : Mathematics

(Functional Analysis II)

(Old)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following questions : **14**

(1) If T is a positive operator on H, then prove that I + T is non-singular.

(2) Prove that $(T + S)^* = T^* + S^*$ and $(TS)^* = S^*T^*$ for each T, S in B(H).

OR

(1) Let H be a Hilbert space, and let f be an arbitrary functional in H* then prove that there is a unique vector y in H such that $f(x) = (x, y)$ for all x in H.

(2) Let $A \in B(H)$ such that $(Ax, x) = 0$ for all x in H, prove that A = 0.

(B) Attempt any **Four** : **4**

(1) Prove that the mapping $y \to f_y$ is norm-preserving.

(2) Define Hilbert space. Give an example of an incomplete inner product space.

(3) Define Isometry. Prove that every isometry is one one.

(4) Give an example of a normal operator that is not unitary.

(5) Prove or disprove: The product of two self-adjoint operators is self-adjoint.

(6) Give a normal operator that is not self-adjoint.

2. (A) Answer the following questions : **14**

(1) Let $T \in B(H)$. Show that T is normal iff $\|T^*x\| = \|Tx\|$ for each $x \in H$.

(2) If P and Q are projections on M and N respectively, show that P + Q is a projection iff PQ = 0.

OR

(1) Show that the unitary operators on H form a group.

(2) If P is a projection on H with range M. Show that $x \in M$ iff $P(x) = x$ iff $\|P(x)\| = \|x\|$.

(B) Attempt any **Four** : **4**

(1) Give an example of a projection on $\mathbb{R}^3$.

(2) Give an example of an operator on $\mathbb{R}^2$ that does not have eigen value.

(3) Define positive operator. Show that sum of two positive operator is positive.

(4) What is the dimension of $\mathbb{C}^2$ over $\mathbb{R}$?

(5) Give a basis of $\mathbb{R}^n$ over $\mathbb{R}$.

3. (A) Answer the following questions : **14**

(1) Define the approximate Eigen spectrum $\sigma_a(A)$. State and prove a characterization of approximate Eigen values.

(2) If X is a Banach space then prove that the set G of all the invertible elements in BL(X) is open.

OR

(1) State and prove Gelfand-Mazur theorem.

(2) If X is a Banach space then prove that the set S of all the singular elements in BL(X) is closed.

(B) Attempt any **Three** : **3**

(1) Prove or disprove : Every operator on $\mathbb{R}^3$ has an eigen-value.

(2) Find the matrix corresponding to the operator T : $(x, y) = (x + 2y, 3x + 4y)$.

(3) Consider the operator T : $(x, y) = (x + 2y, 2x + y)$. Is T invertible ? Justify.

(4) Can we have an operator $T \in B(H)$ such that $\sigma(T) = \mathbb{C}$?

(5) Define the spectral radius $r_a(T)$.

4. (A) Answer the following questions : **14**

(1) Prove that every functional $f \in X^*$ is a compact map.

(2) If A is a compact operator on an infinite dimensional normed linear space X then prove that $0 \in \sigma_a(A)$.

OR

(1) Prove that $F \in BL(X, Y)$ is compact if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(2) If A, B $\in$ CL(X, Y), prove that A + B and $\alpha A \in CL(X, Y)$. (Here α is a scalar.)

(B) Attempt any **Three** : **3**

(1) Prove that very 2×2 real matrix determines a compact map on $\mathbb{R}^2$.

(2) True or false : The right shift operator on the Hilbert space l^2 is compact.

(3) Find an operator A such that $\sigma(A) = \{i\}$.

(4) Give an example of a bounded linear map that is not compact.

(5) Can we find a compact operator A such that $\sigma(A) = \mathbb{C}$? Explain.

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ML-130

March-2019

M.Sc., Sem.-IV

507 : Mathematics

(Differential Geometry)

(New)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) (i) At what angles do the hyperbolas $xy = c_1$, $x^2 - y^2 = c_2$ intersect ? **7**

(ii) Show that the curvature and torsion of the helix $x = \cos t$, $y = \sin t$, $z = t$ are constant. **7**

OR

(i) Show that if the tangents to a curve are parallel to a certain plane, then the curve is a plane curve.

(ii) Find the length of the segment $0 \le t \le 2\pi$ of the cycloid $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$.

(B) Do any **four** : **4**

(i) Sketch (roughly) a family of curves and its envelope.

(ii) Sketch (roughly) a tractrix.

(iii) Write down (without proof) the Frenet formulas.

(iv) Write down (without proof) the curvature of a circle of radius R.

(v) Give (without proof) the equation of a parabola of the form $y = x^2 + ax + b$ which is tangent to the circle $x^2 + y^2 = 1$ at the point $(0, -1)$.

(vi) Give a parametric equation of a circle with centre (1, 2) and radius 3.

2. (A) (i) Show that all the tangent planes to the surface $z = x\, f\left(\frac{y}{x}\right)$ pass through the origin of the co-ordinates. **7**

(ii) Find the equation of the osculating paraboloid to an ellipsoid $\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{5^2} = 1$ at the point (0, 0, 5). **7**

OR

(i) Show that the surfaces $x^2 + y^2 + z^2 = ax$, $x^2 + y^2 + z^2 = bx$, $x^2 + y^2 + z^2 = cx$ intersect at right angles.

(ii) Make up the equation of the surface formed by straight lines passing through a point p = (a, b, c) and intersecting a curve r = r(u) (conical surface).

(B) Do any **four** : **4**

(i) Give the osculating paraboloid of the surface $z = x^2 + y^2 + y^3$ at (0, 0). (Do not prove)

(ii) Give the tangent plane to $z = x^2 + y^3$ at (0, 0). (Do not prove)

(iii) Give a parametrization of the unit sphere. (Do not prove)

(iv) Sketch (roughly) a cylindrical surface.

(v) True or false ? There are elliptic points on a closed surface. (Do not prove)

(vi) True or false ? If a smooth surface has only one point in common with a plane, then the plane is the tangent plane to the surface at that point. (Do not prove)

3. (A) (i) Find the area of the quadrilateral bounded by u = 0, u = 1, v = 0, v = 1 on the helicoid $x = u \cos v$, $y = u \sin v$, $z = v$. **7**

(ii) Find asymptotic lines on the catenoid **7**

$x = \cos h\, u \cos v$, $y = \cos h\, u \sin v$, $z = u$

OR

(i) Find the first fundamental form of the surface of revolution $x = \phi(u) \cos v$, $y = \phi(u) \sin v$, $z = \chi(u)$.

(ii) Prove that the co-ordinate u-curves, v-curves on a translation surface $r(u, v) = U(u) + V(v)$ are conjugate.

(B) Do any **three** : **3**

(i) Sketch (roughly) a loxodrome on a sphere.

(ii) Sketch (roughly) a surface of constant Gaussian curvature = –2.

(iii) State (without proof) the Euler formula.

(iv) Sketch (roughly) the Dupin indicatrix at an elliptic point.

(v) Give an example of a non-identity conformal mapping from the xy-plane to itself.

4. (A) (i) Prove that conical surfaces are locally isometric to a plane. **7**

(ii) Show that an asymptotic geodesic line is a straight line. **7**

OR

(i) Show that if a geodesic line is a line of curvature, then it lies in a plane.

(ii) Prove that the sum of the angles of a geodetic triangle on a surface of positive Gaussian curvature is greater than π.

(B) Do any **three** : **3**

(i) State (without proof) the Gauss-Bonnet theorem for closed surfaces.

(ii) What is the Euler characteristics of a torus ? (Do not prove)

(iii) State (without proof) the extremal property of geodesics.

(iv) Describe (without proof) the geodesics on a sphere.

(v) If a curve $r = r(t)$ lies on a surface, give a formula for its geodesic curvature. (Do not prove)
