Seat No. : _______________

AD-144

April-2019

M.Sc., Sem.-II

409 : Mathematics
(Complex Analysis-II)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (a) (i) If z_1 is a point inside the circle of convergence $| z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ then that series must be uniformly convergent in the closed disk $| z - z_0 | \le R_1$, where $R_1 = |z_1 - z_0|$. **14**

(ii) Show that the sum of the series

$$x^2 + \frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)} + \frac{x^2}{(1 + x^2)^3} + ...$$

is discontinuous at $x = 0$. Also, Obtain the Laurent series expansion and radius of convergence of the function $\text{Ln}\left[\frac{z}{z-1}\right]$, $z_0 = 0$ valid in the region $| z | >1$.

OR

(b) (i) Let C denote any contour interior to the circle of convergence of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, and let g(z) be any function that is continuous on C. Then show that the series formed by multiplying each term of the power series by g(z) can be integrated term by term over C. **14**

(ii) If $| w |< 1/2$, Show that

(a) $| \text{Ln} (l - w) | < 2 | w |$ and

(b) $| \text{Ln} (1 - w) + w| \le | w |^2$.

(c) Attempt any **FOUR** : **4**

(i) If f is analytic everywhere in the finite complex plane except at z_0 then the Laurent's series expansion is valid in the disk ________.

(ii) The radius of convergence of the series $\sum_{n=0}^{\infty} \left(\frac{n\sqrt{2}+i}{1+2in}\right)(z)^n$ is ________.

(iii) The region of convergence of $f(z) = 1 / (e^z - 1)$ about $z_0 = 0$ is ________.

(iv) For what value of z, does the series $\sum_{n=0}^{\infty} \frac{1}{(1+z^2)^n}$ converge, and find its sum.

(v) Find Taylor's series expansion of $f(z) = 1 / [(z+1)(z+2)^2]$ about the point $z_0 = 1$.

(vi) State the definition of uniform convergence of power series.

2. (a) (i) State and prove Cauchy residue theorem. **14**

(ii) Discuss the singularity of the function $f(z) = \frac{(z^2-1)(z-2)^3}{(\sin \pi z)^3}$ in the extended complex plane.

OR

(b) (i) Suppose that Z_o is an essential singularity of function f and let w_o be any complex number. Then prove that for any positive ε, the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some point z in each deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0. **14**

(ii) Find the residue of the function $f(z) = \cos\left(\frac{z^2+6z+5}{z+2}\right)$ at $z = -2$.

(c) Attempt any **FOUR** : **4**

(i) An analytic function is bounded in some neighbourhood of a removable singular point. (True/False)

(ii) If f(z) be an entire function and not a constant. Then the function f(z) has ________ or has ________ singularity at $z = \infty$.

(iii) Classify the singularity of the $f(z) = \frac{\mathrm{Ln}(1+z^m)}{z^n}$, m and n are positive integers in the finite complex plane.

(iv) Classify the singular point z = 0 of the function $f(z) = \frac{e^z}{z - \sin z}$.

(v) Give a simple example of a function which has a non-removable singularity at z = 0 and whose residue at that point is zero.

(vi) Find the residue of the function $h(z) = \frac{f'(z)}{f(z)}$ at z = a if z = a is a pole of order n of f(z).

3. (a) (i) Show that if function is even then Cauchy principal value of the integral is equal to improper integral and evaluate the integral $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2}$, (a > 0). **14**

(ii) Evaluate the integral $I = \int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx$.

OR

(b) (i) Show that if improper integral exists then Cauchy principal value of the integral exists but not vice versa. **14**

(ii) Evaluate the integral $I = \int_{0}^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$.

(c) Attempt any **THREE** : **3**

(i) State Jordan's inequality.

(ii) The value if the integral $I = \int_{0}^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}$, (a > b> 0) is ______.

(iii) The residue at all the singular points of $f(z) = \frac{z^2}{(z^n - 1)}$, n is any positive integer is ________.

(iv) The value if the integral $I = \oint_{C} \frac{dz}{z^4 + 1}$ C : | z – 1 | = 1 is ______.

4. (a) (i) Suppose that function f(z) has a simple pole at a point $z = x_0$ on the real axis, with Laurent series expansion in the punctured disk $0 < |z - x_0| < R_2$ and with residue B_0. Let C_ρ denotes the upper half of the circle $|z - x_0| = \rho$, where $\rho < R_2$ and the clockwise direction is taken then prove that $\lim_{\rho \to 0} \oint_{C_\rho} f(z)\, dz = -B_0 \pi i$. **14**

(ii) Evaluate the integral $I = \int_{0}^{\infty} \frac{x^a}{(1 + x)}\, dx$, 0 < a < 1.

OR

(b) (i) State and prove Argument principle. **14**

(ii) Prove that the equation $z^3 + iz + 1 = 0$ has one root in each of the first quadrant, second quadrant and four quadrants.

(c) Attempt any **THREE** : **3**

(i) State Conformal mapping.

(ii) The fixed point of inverse mapping is ______.

(iii) The bilinear transformation $w = \frac{iz + 2}{4z + i}$ transforms the real axis in the z-plane onto a _______ in the w-plane. Justify it.

(iv) A bilinear transformation more than two fixed point is _______.