Seat No. : _______________

AB-159

April-2019

M.Sc., Sem.-II

407 : Mathematics
(Metric Spaces)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Answer the following questions : **14**

(1) State and prove the theorem about the structure of open sets in $\mathbb{R}$.

(2) Let X be a metric space and $A \subset X$. Define the interior int(A) of A and the closure $\bar{A}$ of A. Find int(A) and $\bar{A}$ for the following sets :

(i) $A = \mathbb{Z}$ (ii) $A = \{(x, y) \in \mathbb{R}^2 / x \notin \mathbb{Z}, y \notin \mathbb{Z}\}$.

OR

(1) Define equivalent metrics. If metrics d and d' are equivalent on X, show that each open ball in X relative to d contains an open ball relative to d' and vice versa.

(2) Let X be a metric space and $E \subset X$. Define the limit point and cluster point of E. Find the limit poins and cluster points of the following sets:

(i) $A = \mathbb{Q}(\sqrt{5})$ (ii) $A = \{(x, y) \in R^2 / x \in \mathbb{Z}, y \in \mathbb{Z}\}$.

(B) Attempt any **Four** : **4**

(1) Show that in the discrete metric space (X, d), each subset of X is open and closed.

(2) Define the norm $\| f \|_1$ on C[0, 1]. Give geometric meaning of $\| f \|_1$.

(3) Let A be any finite set in metric space (X, d). Show that A is closed.

(4) If A is open and B is closed in metric space (X, d), prove that A – B is open.

(5) When do we say that $\mathcal{T}$ is the topology determined by metric d ? Explain.

(6) For any $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, show that $\| x \|_\infty \leq \| x \|_2 \leq \| x \|_1$.

2. (A) Answer the following questions : **14**

(1) Prove that the limit of a convergent sequence in a metric space is unique.

(2) Let A be a subset of a space X. Define the boundary of A. Find the boundary of the following sets:

(i) $A = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. (ii) $A = \{(x, y) \in \mathbb{R}^2 / x \in \mathbb{Q}, y \in \mathbb{Q}\}$.

OR

(1) Prove that a subset E of a metric space (X, d) is closed iff E contains all its limit points.

(2) Define Dense set. Show that $D \subset \mathbb{R}$ is dense in $\mathbb{R}$ iff each real number is a limit point of D.

(B) Attempt any **four** : **4**

(1) In $\mathbb{R}$, prove that $x_n \to x$ and $y_n \to y$ implies that $x_n y_n \to xy$.

(2) Define the Cauchy sequence. Determine all the Cauchy sequences in the discrete metric space X.

(3) Define complete metric space. Give an example of a complete metric space.

(4) Consider the sequence $\{\log n\}$ in $\mathbb{R}$. Is this a Cauchy sequence ? Explain.

(5) State (without proof) Weierstrass Approximation Theorem.

3. (A) Answer the following questions : **14**

(1) Let X, Y be metric spaces. Let $f : X \to Y$ be a function. Show that the following are equivalent :

(i) f is continuous at x.

(ii) Given any $\in > 0$, there is $\delta > 0$ such that $d(f(x), f(y)) < \in$ whenever $d(x, y) < \delta$.

(iii) Given any open set V containing f(x) in Y, there is an open set U containing x such that $f(U) \subset V$.

(2) Show that addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $+(a, b) = a + b$ and multiplication $: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\cdot(a, b) = a \cdot b$ are continuous functions.

OR

(1) Let (X, d) be a metric space. Prove that the set $C(X, \mathbb{R})$ of all real-valued continuous functions on X is vector space with respect to pointwise operations.

(2) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = g(x)$ for each $x \in \mathbb{Q}$. Then prove that $f(x) = g(x)$ for each $x \in \mathbb{R}$.

(B) Attempt any **three** : **3**

(1) Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.

(2) Prove that $\{(x, y) \in \mathbb{R}^2 / \sin x + \cos xy + e^{x^2+y^2} > 1\}$ is open in $\mathbb{R}^2$.

(3) Define d(A, B). Show that there are disjoint, closed sets A and B such that $d(A, B) = 0$.

(4) State (without proof) Urysohn's lemma.

(5) Show that a circle and an ellipse in $\mathbb{R}^2$ are homeomorphic.

4. (A) Answer the following questions : **14**

(1) Prove that any continuous function from a compact metric space to any other metric space is uniformly continuous.

(2) Prove that a topological space X is connected iff every continuous function $f : X \rightarrow \{-1, 1\}$ is constant.

OR

(1) State and prove the Intermediate Value Theorem.

(2) Prove that any compact subset of a metric space is closed and bounded.

(B) Attempt any **three** : **3**

(1) Show that $\mathbb{R}$ with respect to the usual metric d is not compact.

(2) True or False: Every countable set is compact. Justify.

(3) Give an example of a countable disconnected set. Explain.

(4) Show that $\mathbb{R}$ is homeomorphic to $(-1, 1)$.

(5) Show that connectedness is a topological property.