1104E078

Candidate's Seat No :_________

M.Sc. (Sem.-IV) Examination

507

Mathematics

Time : 3 Hours] **April-2017** **[Max. Marks : 70**

MAT507 Functional Analysis-II

1. (a) Attempt any ONE. **7**

   (i) Show that the adjoint operation $T \to T^*$ is a one to one onto as a mapping of B(H) into itself.

   (ii) Let H be a Hilbert space. Define $f_y(x) = (x.y)$ for all x in H. Show that $f_y \in H^*$. Prove that the mapping $y \to f_y$ from H into H^* is surjective.

   (b) Attempt any Two. **4**

   (i) Let $A \in B(H)$ such that $(Ax, x) = 0$ for all x in H, prove that $A = 0$.

   (ii) Let $H = \mathbb{R}^2. K = \mathbb{R}$ and $A \in BL(H)$ is given by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that A is normal if and only if $b = c$ or else $b = -c.\ a = d$.

   (iii) Show that the set of all self-adjoint operators in B(H) is a non-empty and closed subset of B(H).

   (c) Answer very briefly. **3**

   (i) Prove or disprove: Every isometry is invertible.

   (ii) Prove that the mapping $y \to f_y$ is norm-preserving.

   (iii) Show that among all the norms $||.||_p, 1 \leq p \leq \infty$, on $\mathbb{K}^n (n \geq 2)$, only the norm $||.||_2$ is induced by an inner product.

2. (a) Attempt any ONE. **7**

   (i) Show that an isometric linear operator on H which is not unitary maps the Hilbert space H onto a proper closed subspace of H.

   (ii) Define unitary operator on H. Show that $T \in B(H)$ is unitary $\Leftrightarrow$T($\{e_i\}$) is a complete orthonormal set whenever $\{e_i\}$ is.

   (b) Attempt any TWO. **4**

   (i) If P and Q are projections on M and N respectively, under what condition(s) does P-Q become a projection ? What is the range of P-Q?

   (ii) If P is a projection show that R(P),the range of P and and N(P), the null space of P are closed.

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(iii) If P and Q are projections on M and N respectively, prove that

$P \leq Q \Leftrightarrow M \subset N \Leftrightarrow PQ = P$

(c) Answer very briefly. **3**

(i) Define the terms: (i) M is invariant under T. (ii) M reduces T.

(ii) Prove or disprove: If A and B are positive operators then AB is positive.

(iii) Give an example of an operator on l^2 that does not have eigenvalue.

3. (a) Attempt any ONE. **7**

(i) Prove that two matrices in A_n are similar if and only if they are the matrices of a single operator on H relative to different bases.

(ii) State and prove the finite dimensional spectral theorem.

(b) Attempt any TWO. **4**

(i) If T is normal operator on a finite dimensional space H , prove that T^* is a polynomial in T.

(ii) If A is non-singular, prove that $\sigma(ATA^{-1}) = \sigma(T)$

(iii) If $T^3 = 0$ then prove that $\sigma(T) = \{0\}$.

(c) Answer very briefly. **3**

(i) Show that a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ must have non-empty spectrum.

(ii) Find the matrix corresponding to $T(x, y) = (x + y, x - 2y)$.
Is T non-singular?

(iii) Find the spectrum of $A(x, y) = (y, x)$ on $\mathbb{R}^2$.

4. (a) Attempt any ONE. **7**

(i) If X is a Banach space and G denote the set of invertible operators in $BL(X)$, prove that G is open in $BL(X)$. Further, prove that the map $x \to x^{-1}$ is continuous.

(ii) Let X be a Banach space and $A \in BL(X)$. Prove that A is invertible $\Leftrightarrow$ A is bounded below and the range of A is dense in X.

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(b) Attempt any TWO. **4**

(i) Find the spectrum of $A(x) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$ where $x \in l^p$.

(ii) Give a characterization of the approximate eigen spectrum $\sigma_a(A)$.

(iii) Define the spectral radius of A. Find the spectral radius of the multiplication operator $A : C[0,1] \to C[0,1]$ defined by $A(x) = x_0 x$, where $x_0(t) = 2t - 1$ for t$\in [0,1]$.

(c) Answer very briefly. **3**

(i) Find an operator A such that $r_\sigma(A)$ is strictly less than $lim_{n\to\infty}||A^n||^{1/n}$.

(ii) Is the operator A(x,y,z)=(x-1, y, z) on $\mathbb{R}^3$ invertible? If yes, find the inverse.

(iii) State (only) : (i) Gelfand-Mazur Theorem (ii) Spectral radius formula.

5. (a) Attempt any ONE. **7**

(i) Prove that $F \in BL(X,Y)$ is compact if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(ii) Let X be a normed linear space and $A \in CL(X)$. Prove that $\sigma_e(A)$ and $\sigma(A)$ are countable sets and have 0 as the only possible limit point.

(b) Attempt any TWO. **4**

(i) Prove that every functional $f \in X^*$ is a compact map.

(ii) If $A, B \in CL(X,Y)$, prove that $A + B \in CL(X,Y)$.

(iii) Prove or disprove: The right shift operator on l^2 is a compact map.

(c) Answer very briefly. **3**

(i) True or false: The map $A : l^2 \to l^2$ defined by $A(x) = 3x$ is a compact operator. Justify.

(ii) Prove or disprove: The linear map T: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (y.2x)$ is compact.

(iii) Can we find a compact operator A such that $\sigma_a(A) = (0,1)$? Justify.
