1104E078

Candidate's Seat No:	
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M.Sc. (Sem.-IV) Examination 507

Mathematics

Time: 3 Hours

April-2017

[Max. Marks: 70

MAT507 Functional Analysis-II

1. (a) Attempt any ONE.

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- (i) Show that the adjoint operation $T \to T^*$ is a one to one onto as a mapping of B(H) into itself.
- (ii) Let H be a Hilbert space. Define $f_y(x) = (x, y)$ for all x in H. Show that $f_y \in H^*$. Prove that the mapping $y \to f_y$ from H into H^* is surjective.
- (b) Attempt any Two.

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- (i) Let $A \in B(H)$ such that (Ax, x) = 0 for all x in H, prove that A = 0.
- (ii) Let $H = \mathbb{R}^2$. $K = \mathbb{R}$ and $A \in BL(H)$ is given by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that A is normal if and only if b = c or else b = -c. a = d.
- (iii) Show that the set of all self-adjoint operators in B(H) is a non-empty and closed subset of B(H).
- (c) Answer very briefly.

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- (i) Prove or disprove: Every isometry is invertible.
- (ii) Prove that the mapping $y \to f_y$ is norm-preserving.
- (iii) Show that among all the norms $||.||_p, 1 \le p \le \infty$, on $\mathbb{K}^n (n \ge 2)$, only the norm $||.||_2$ is induced by an inner product.
- 2. (a) Attempt any ONE.

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- (i) Show that an isometric linear operator on H which is not unitary maps the Hilbert space H onto a proper closed subspace of H.
- (ii) Define unitary operator on H. Show that $T \in B(H)$ is unitary $\Leftrightarrow T(\{e_i\})$ is a complete orthonormal set whenever $\{e_i\}$ is.
- (b) Attempt any TWO.

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- (i) If P and Q are projections on M and N respectively, under what condition(s) does P-Q become a projection? What is the range of P-Q?
- (ii) If P is a projection show that R(P), the range of P and and N(P), the null space of P are closed.

E078-2

- (iii) If P and Q are projections on M and N respectively, prove that $P \leq Q \Leftrightarrow M \subset N \Leftrightarrow PQ = P$
- (c) Answer very briefly.

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- (i) Define the terms: (i) M is invariant under T. (ii) M reduces T.
- (ii) Prove or disprove: If A and B are positive operators then AB is positive.
- (iii) Give an example of an operator on l^2 that does not have eigenvalue.
- 3. (a) Attempt any ONE.

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- (i) Prove that two matrices in A_n are similar if and only if they are the matrices of a single operator on H relative to different bases.
- (ii) State and prove the finite dimensional spectral theorem.
- (b) Attempt any TWO.

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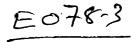
- (i) If T is normal operator on a finite dimensional space H , prove that T^* is a polynomial in T.
- (ii) If A is non-singular, prove that $\sigma(ATA^{-1}) = \sigma(T)$
- (iii) If $T^3 = 0$ then prove that $\sigma(T) = \{0\}$.
- (c) Answer very briefly.

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- (i) Show that a linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ must have non-empty spectrum.
- (ii) Find the matrix corresponding to T(x,y)=(x+y,x-2y). Is T non-singular?
- (iii) Find the spectrum of A(x,y) = (y,x) on \mathbb{R}^2 .
- 4. (a) Attempt any ONE.

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- (i) If X is a Banach space and G denote the set of invertible operators in BL(X), prove that G is open in BL(X). Further, prove that the map $x \to x^{-1}$ is continuous.
- (ii) Let X be a Banach space and $A \in BL(X)$. Prove that A is invertible \Leftrightarrow A is bounded below and the range of A is dense in X.



- (b) Attempt any TWO.
 - (i) Find the spectrum of $A(x)=(0,x_1,\frac{x_2}{2},\frac{x_3}{3},\dots)$ where $x\in l^p$.
 - (ii) Give a characterization of the approximate eigen spectrum $\sigma_a(A)$.
 - (iii) Define the spectral radius of A. Find the spectral radius of the multiplication operator $A: C[0,1] \to C[0,1]$ defined by $A(x) = x_0x$, where $x_0(t) = 2t 1$ for $t \in [0,1]$.
- (c) Answer very briefly.

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- (i) Find an operator A such that $r_{\sigma}(A)$ is strictly less than $\lim_{n\to\infty} ||A^n||^{1/n}$.
- (ii) Is the operator A(x,y,z)=(x-1, y, z) on \mathbb{R}^3 invertible? If yes, find the inverse.
- (iii) State (only): (i) Gelfand-Mazur Theorem (ii) Spectral radius formula.
- 5. (a) Attempt any ONE.

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- (i) Prove that $F \in BL(X,Y)$ is compact if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.
- (ii) Let X be a normed linear space and $A \in CL(X)$. Prove that $\sigma_e(A)$ and $\sigma(A)$ are countable sets and have 0 as the only possible limit point.
- (b) Attempt any TWO.

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- (i) Prove that every functional $f \in X^*$ is a compact map.
- (ii) If $A, B \in CL(X, Y)$, prove that $A + B \in CL(X, Y)$.
- (iii) Prove or disprove: The right shift operator on l^2 is a compact map.
- (c) Answer very briefly.

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- (i) True or false: The map $A:l^2\to l^2$ defined by A(x)=3x is a compact operator. Justify.
- (ii) Prove or disprove: The linear map $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by T(x,y) = (y,2x) is compact.
- (iii) Can we find a compact operator A such that $\sigma_a(A) = (0,1)$? Justify.