1705E649

Candidate's Seat No :

M.Sc. (Sem.-III) Examination
501 Mathematics
May-2017

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any ONE. 7

(i) If E is a linear transformation on a linear space L, then prove that E is idempotent $\Leftrightarrow$ there exist subspaces M and N of L such that L= $M \bigoplus N$ and E is the projection on M along N.

(ii) Let L be a non-zero linear space. If $B_1 = \{e_i\}$ and $B_2 = \{f_j\}$ be two bases for L, then prove that B_1 and B_2 have the same number of elements (that is, the same cardinal number).

(b) Attempt any Two. 4

(i) Let T be a linear transformation on a linear space L. Prove that T is non-singular $\Leftrightarrow$ T(B) is a basis for L whenever B is.

(ii) Show that the set $\{1, x, x^2, x^3 ...\}$ is a linearly independent subset of $C[0, 1]$

(iii) If $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$, then is it true that $\{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$ is also a basis? Justify.

(c) Answer very briefly. 3

(i) Give two linear spaces over $\mathbb{R}$ of dimension 5.

(ii) Give examples of three proper subspaces of $\mathbb{R}^3$.

(iii) Let L be a non-zero finite dimensional linear space of dimension n. Show that every set of n+1 vectors in L is linearly dependent.

2. (a) Attempt any ONE. 7

(i) If M is a closed linear subspace a normed linear space N then show that the quotient N/M is also a normed linear space.

(ii) If T is a linear transformation from a normed linear space X to a normed linear space Y then prove that T is continuous $\Leftrightarrow$ T is continuous at 0 $\Leftrightarrow$ there exist a real number $K \geq 0$ such that $||T(x)|| \leq K||x||$ for all x.

(P.T.O)

(b) Attempt any TWO. **4**

(i) Give an example of a discontinuous linear transfromation.

(ii) If T is a linear transformation from a finite dimensional normed linear space X to any normed linear space Y then prove that T is continuous.

(iii) Define the equivalence of two norms $||.||$ and $||.||'$. Give an illustration.

(c) Answer very briefly. **3**

(i) If T is a continuous (bounded) linear transformation then prove that T is uniformly continuous.

(ii) Is the map T: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by T(x,y)= (y,x) linear? continuous? Justify.

(iii) Define isometric isomorphism.

3. (a) Attempt any ONE. **7**

(i) State and prove Hahn- Banach theorem.

(ii) Show that a normed linear space N can be regarded as a part of N^{**} without altering any of its structure as a normed linear space.

(b) Attempt any TWO. **4**

(i) Define the conjugate space of a normed linear sapce. If N is finite dimensional then show that its conjugate space N^* is also finite dimensional.

(ii) If y is a non-zero vector in a normed linear space N then show that there is functional f on N such that $f(y) = ||y||$ and $||f|| = 1$.

(iii) Give an example of a functional on $C[0, 1]$.

(c) Answer very briefly. **3**

(i) Define reflexive normed linear space. Give an illustartion.

(ii) Give the conjugate space of normed linear spaces c_0 and l_1.

(iii) Is it true that the conjugate space of any normed linear space is complete? why?

4. (a) Attempt any ONE. **7**

(i) State and prove Open mapping theorem,

(ii) State and prove Closed graph theorem.

E 649-3

(b) Attempt any TWO. 4

(i) If P is a projection on a Banach space B then show that its range and the null space are closed linear subspaces of B.

(ii) Prove that a non-empty subset A of a normed linear space N is bounded $\Leftrightarrow$ f(A) is a bounded set of numbers for each f in N^*.

(iii) If T is an operator on a Banach space B, show that T has an inverse $\Leftrightarrow$ T^* has an inverse.

(c) Answer very briefly. 3

(i) Give an example of a projection on $\mathbb{R}^3$. Give its range and the null space.

(ii) True or False: The map T(x,y,z)=(x-1, y, z) on $\mathbb{R}^3$ is a projection.

(iii) Show that $(ST)^* = T^*S^*$.

5. (a) Attempt any ONE. 7

(i) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

(ii) State and prove Schwarz' inquality. Using this prove that the inner product is jointly continuous.

(b) Attempt any TWO. 4

(i) Give an example of an inner product space that is not a Hilbert space.

(ii) Define the orthogonal complement $S^{\perp}$ of a subset S of Hilbert space H. Show that $S^{\perp}$ is a closed linear subspace of H.

(iii) If $A = \{x_1, x_2, ..., x_n\}$ is a set of non-zero orthogonal vectors in H then prove that A is linearly independent.

(c) Answer very briefly. 3

(i) State and prove parallelogram law.

(ii) Define orthonormal basis in Hilbert space.

(iii) True or False : Every Hilbert space is reflexive.

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1905E714

Candidate's Seat No : ____

M.Sc. (Sem.-III) Examination
503 EA Mathematics
May-2017

Time : 3 Hours] **[Max. Marks : 70**

Q.1 (a) Attempt any one. 7

(i) Prove that there is an infinite number of primes.

(ii) Find integers x, y, z such that $35x + 55y + 77z = 1$.

(b) Attempt any two. 4

(i) If p is a prime number and p/ab, prove that p/a or p/b.

(ii) Prove that Fermat numbers are all relatively prime to each other.

(iii) Prove that the sum of the squares of two odd integers cannot be a perfect square.

(c) Answer very briefly. 3

(i) State fundamental theorem of arithmetic.

(ii) Show that any integer of the form $6n + 5$ is also of the form $3k + 2$, but not conversely.

(iii) Find all pairs of primes p and q satisfying $p - q = 3$.

Q.2 (a) Attempt any one. 7

(i) Prove that every even perfect number is of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ $(k > 1)$is a prime.

(ii) State and prove the Möbius inversion formula.

(b) Attempt any two. 4

(i) Prove that $[x] + [y] \leq [x + y]$, where x and y are real numbers.

(ii) Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square.

(iii) If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, prove that

$$\sum_{d|n} \mu(d)\phi(d) = (2 - p_1)(2 - p_2) \dots (2 - p_r).$$

[P.T.O.

(c) Answer very briefly. **3**

(i) Find the highest power of 5 dividing 5000!.

(ii) Calculate $\phi(2017)$.

(iii) Calculate $\sigma(957)$.

Q.3 (a) Attempt any one. **7**

(i) Using the theory of indices, solve $4x^9 \equiv 7 \pmod{13}$.

(ii) State and prove Wilson's theorem.

(b) Attempt any two. **4**

(i) Solve: $x \equiv 5 \pmod 6, x \equiv 4 \pmod{11}, x \equiv 3 \pmod{17}$.

(ii) Solve: $5x \equiv 2 \pmod{26}$.

(iii) Show that 17 divides $11^{104} + 1$.

(c) Answer very briefly. **3**

(i) Find all primitive roots of 15.

(ii) What is the order of the integer 5 modulo 23?

(iii) State Euler's theorem.

Q.4 (a) Attempt any one. **7**

(i) Determine the general solution of $172x + 20y = 1000$ by means of simple continued fractions.

(ii) Find the infinite continued fraction representation of $\frac{1+\sqrt{13}}{2}$.

(b) Attempt any two. **4**

(i) Evaluate $[0; \overline{1, 2, 3}]$.

(ii) Evaluate $[2; \overline{1, 1, 1, 4}]$

(iii) Obtain all primitive Pythagorean triples of the form $60, y, z$.

(c) Answer very briefly. **3**

(i) If x, y, z is a primitive Pythagorean triple. prove that exactly one of the integers x or y is divisible by 3.

(ii) Express $\frac{51}{19}$ as finite simple continued fraction.

(iii) What is the fundamental solution of $x^2 - 18y^2 = 1$?

Q.5 (a) Attempt any one. 7

(i) Determine all algebraic integers of the field $\mathbb{Q}(\sqrt{m})$ where m is a square-free rational integer, positive or negative but not equal to 1.

(ii) Show that the field $\mathbb{Q}(\sqrt{-3})$ is Euclidean.

(b) Attempt any two. 4

(i) Prove that if γ is an integer in $\mathbb{Q}(\sqrt{m})$, then $N(\gamma) = \pm 1$ if and only if γ is a unit.

(ii) Find the minimal polynomial of $1 + \sqrt{2} + \sqrt{3}$.

(iii) If α is any algebraic number, prove that there is a rational integer b such that $b\alpha$ is an algebraic integer.

(c) Answer very briefly. 3

(i) Prove that reciprocal of a unit is a unit.

(ii) Prove that $N(\alpha) = 0$ if and only if $\alpha = 0$.

(iii) Give an example of prime in $\mathbb{Q}(i)$.

1805E673 Candidate's Seat No ________

M.Sc. (Sem.-III) Examination
502 Mathematics
May-2017

Time : 3 Hours] [Max. Marks : 70

1 (A). Define a prime ideal of a commutative ring R with unity. [7]
Define a maximal ideal of a commutative ring R with unity.
Let R be a commutative ring with unity and let A be an ideal of R. Show that R/A is an integral domain if and only if A is a prime ideal.

OR

1 (A) Define a ring homomorphism from a ring R to a ring S. [7]
Define the kernel of a ring homomorphism.
Suppose $\phi : R \to S$ is a homomorphism.
Show that $Ker\,\phi$ is an ideal of R.

1 (B). Answer any two. [4]

(i) Suppose D is an integral domain. Show that $D[x]$ is an integral domain.

(ii) Let $2x+1$ and x^2+2 be polynomials in $\mathbb{Z}_5[x]$. Find the remainder upon dividing x^2+2 by $2x+1$.

(iii) Find a polynomial with integer coefficients which has $\frac{1}{5}$ and $-\frac{1}{3}$ as zeros.

1 (C). Answer all. [3]

(i) Let $f(x) = (x-2)^2(x-1)(x^2-1)$ be a polynomial in $\mathbb{R}[x]$. Then 1 is a zero of multiplicity k. Find k.

(ii) Give an example (without proof) of a prime ideal in $\mathbb{Z}$.

(P.T.O.)

E673-2

1(C)(iii) Let $x^2+1 \in \mathbb{Z}_5[x]$. Find all the zeros of x^2+1 in $\mathbb{Z}_5$.

2(A). State Eisenstein's criterion. [7]
Let p be a prime number.
Show that the polynomial
$\Phi(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$
is irreducible over $\mathbb{Q}$.

OR

2(A). Construct a field with nine elements. [7]

[4]

2(B). Answer any two.

(i) Let $f(x) = x^3+6 \in \mathbb{Z}_7[x]$. Write $f(x)$ as a product of irreducible polynomials over $\mathbb{Z}_7$.

(ii) Is $\frac{2}{3}$ a root of $x^{91}+x^{80}-5=0$?

(iii) Show that $1-i$ is irreducible in $\mathbb{Z}[i]$.

[3]

2(c). Answer all.

(i) Determine the units in $\mathbb{Z}[i]$.

(ii) List all monic polynomials of degree 2 over $\mathbb{Z}_2$ which are irreducible.

(iii) Define a Unique Factorization Domain (UFD).

3 (A). Find the splitting field of x^3-1 over $\mathbb{Q}$. [7]
Find the splitting field of x^4+1 over $\mathbb{Q}$.

OR

3(A). Suppose E is an extension of F of prime degree. Show that, for every a in E, $F(a) = F$ or $F(a) = E$. [7]

Let K be an extension of F. Suppose that E_1 and E_2 are contained in K and are extensions of F. If $[E_1 : F]$ and $[E_2 : F]$ are both prime, show that $E_1 = E_2$ or $E_1 \cap E_2 = F$.

3(B). Answer any two. [4]

(i) Find the minimal polynomial for $\sqrt{-3} + \sqrt{2}$ over $\mathbb{Q}$.

(ii) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

(iii) Let $f(X) = X^4-1 \in \mathbb{Q}[X]$. Does $f(X)$ have a multiple zero?

3(C). Answer all. [3]

(i) Expand the polynomial $(x+1)^3$ in $\mathbb{Z}_3[X]$.

(ii) State (without proof) the Primitive Element Theorem.

(iii) Find the degree $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$.

(P.T.O)

4 (A). State (without proof) a theorem describing all the subfields of the finite field $GF(p^n)$ of p^n elements. [7]

Draw the lattice of subfields of $GF(64)$.

OR

4 (A). State (without proof) a theorem describing all the subfields of the finite field $GF(p^n)$ of p^n elements. [7]

Determine the possible finite fields whose largest proper subfield is $GF(2^5)$.

4 (B). Answer any two. [4]

(i) When is a real number α said to be constructible?

(ii) Suppose c is constructible. What can be said about the degree $[Q(c) : Q]$. (Do not prove).

(iii) Give an example of a number which is not constructible.

4 (C). Answer all. [3]

(i) Prove that 45° is a constructible angle.

(ii) Give a geometrical construction of the length $\sqrt{2}$.

(iii) Find the points of intersection of the circle $x^2 + y^2 = 1$ and the line $y - x = 0$.

5(A). Let $F = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Discuss the lattice of subgroups of $Gal(F/\mathbb{Q})$, and the lattice of subfields of F. [7]

OR

5(A). Let $w = \cos(\frac{2\pi}{7}) + i\sin(\frac{2\pi}{7})$; [7]
Let $F = \mathbb{Q}(w)$. Find the Galois group $Gal(F/\mathbb{Q})$. Discuss its lattice of subgroups and the corresponding lattice of subfields of F.

5(B). Answer any two. [4]

(i) Show that S_3 is solvable.

(ii) Let F be a field and let $f(x) \in F[x]$. Define what is meant by "$f(x)$ is solvable by radicals over F."

(iii) The polynomial $g(x) = 3x^5 - 15x + 5$ has splitting field K and $Gal(K/\mathbb{Q}) \approx S_5$. Is $g(x)$ solvable by radicals?

5(C). Answer all. [3]

(i) Write down the 4^{th} cyclotomic polynomial $\Phi_4[x]$.

(ii) State (without proof) the theorem of Gauss about the constructibity of a regular n-gon.

(iii) Factor $x^6 - 1$ as a product of irreducible polynomials over $\mathbb{Z}$.

2205E795 Candidate's Seat No : ____________

M.Sc. (Sem.-III) Examination
505 Mathematics
May-2017

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any ONE. 7

(i) Let $A = \{(x,y) : x > 0, y > 0, 0 < xy < 3, x < y < 2x\}$, $f(x,y) = y^2$ $g(s,t) = \sqrt{st}e_1 + \sqrt{st}e_2$ for $s > 0, t > 0$. Show that g is univalent. Find $\int_A y^2 dV_2(x,y)$ where A denotes the part of hyperbola $xy = c, 0 < c < 3$ corresponding to the segment s=c, $1 < t < 2$ in B.

(ii) Find the area of $A = \{(x,y)/x^2 \leq y \leq x+2\}$.

(b) Attempt any Two. 4

(i) Evaluate integral$\int_1^0 dx \int_1^0 exp(x+y)dy$..

(ii) Find the area of $\left\{(x,y) : |y| - 1 \leq \sqrt{1-y^2}\right\}$.

(iii) Find the volume of the tetrahedron with vertices $e_1, -e_2, e_3, e_1 + 2e_2 + e_3$.

(c) Answer very briefly. 3

(i) Is the function $f(x) = x - |x|$ bounded? Find its support.

(ii) Show that $f(x) = sin(1/x)$ if $x \neq 0$, f(0)= 1 is integrable over A= [-1, 1].

(iii) Let $A = \{(y,1)/0 \leq y \leq 5\}$. Find $V_2(A)$.

2. (a) Attempt any ONE. 7

(i) Let n=3. Define a multicovector of degree 2. Define the 2-covector e^λ where $\lambda = (1,2)$. Give a basis of $(E_2^3)^*$.

(ii) Define r-linear alternating function. If M is r-linear alternating function then prove that $M(h_1, h_2, .., h_r) = 0$ whenever $(h_1, h_2, ..., h_r)$ is linearly dependent.

(b) Attempt any TWO. 4

(i) Let n=3. Give an example of a 3-linear.alternationg function.

(ii) Find the area of the triangle with vertices $0, 3e_1 + e_2, e_3 - e_2$.

(iii) Define a a frame. Show that $(e_1 - e_2, e_2 - e_3)$ and $(3e_1 - e_2 - 2e_3, 2e_1 - e_2 - e_3)$ are frames for the same vector subspace of E^3.

(P.T.O)

E 795-2

(c) Answer very briefly. **3**

(i) For n=5, find δ^{255}_{525}

(ii) For n=5, simplify: $(e^1 - e^2 + 3e^3) \wedge e^{21}$

(iii) For n=5, simplify $e^2 \wedge (3e^1 - 2e^3) \wedge e^5 \wedge e^3$.

3. (a) Attempt any ONE. **7**

(i) Define a differential form w of degree r on a domain $D \subset E^n$. When is an r-form w said to be of class $C^{(1)}$? Define the exterior differential dw. Find the exterior diffrential of $w = cos(xy^2)dx \wedge dz$. Define a closed form. Show that $w \wedge \xi$ is closed if w and ξ are closed.

(ii) Define the adjoint of r-vector α. Let n=2. Find the adjoint of (a) d(Ndx-Mdy)(b) Mdx+Ndy .

(b) Attempt any TWO. **4**

(i) Define an exact form. Give an illustration of an exact 1-form.

(ii) Let n=3, and $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ be a 2-form Find ω^*.

(iii) Let n=3, and $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ be a 2-form Find $d\omega$.

(c) Answer very briefly. **3**

(i) Show that $curl(f\omega) = fcurl\omega + df \times \omega$

(ii) Define the term: Induced linear transformations.

(iii) Prove that the cross product is not associative.

4. (a) Attempt any ONE. **7**

(i) Define a coordinate system on a non-empty, relatively open subset of an r-manifold. Give an illustration to explain this coordinate system.

(ii) Define a regular transformation g from an r-manifold N into an r-manifold M. Explain each term that occurs in the definition.

E 795-3

(b) Attempt any TWO. **4**

(i) Consider the transformation $g(s,t) = (s+t)e_1 + (s-3t)e_2 + (2t-2s-2)e_3$ on $\Delta = \{(s,t)/0 < s+t < 1, s > 0, t > 0\}$. Find $J_g(s,t)$ and $g(\Delta)$. Is g univalent?

(ii) Let $g(s,t) = ste_1 + se_2 + te_3$, $\Delta = E^2$. Find $J_g(s,t)$ and $g(\Delta)$.

(iii) Define compact manifold. Give an example.

(c) Answer very briefly. **3**

(i) Define r-manifold. Give examples of a 1-manifold and a 2-manifold.

(ii) Define a partition of unity for a compact manifold M.

(iii) Define orientable manifold. Give a simple example.

5\. (a) Attempt any ONE. **7**

(i) Let D be a regular domain, let n=2 , show that $V_2(D) = -\int_{\delta D^+} y dx$.

(ii) State (only) Green's theorem. Using Green's theorem find the area of the closed unit disc in E^2.

(b) Attempt any TWO. **4**

(i) Show that any straightline segment in E^2 has area 0.

(ii) Define a regular domain in E^3.

(iii) Show that the circle in E^2 is a 1-manifold.

(c) Answer very briefly. **3**

(i) Let α_n be the measure of the unit n-ball $\{x/|x| \leq 1\}$. Find the value of α_2 and α_3?

(ii) True or false : If A is a countable subset of $\mathbb{R}$ then $M(A) = 0$.

(iii) Find the area of the triangle with vertices $2e_3, e_1 - e_2 + 2e_3, e_1 + 3e_3$.

2005E759

Candidate's Seat No :___

M.Sc. (Sem.-III) Examination

504 Mathematics

Time : 3 Hours] **May-2017** **[Max. Marks : 70**

1(a) Attempt any **ONE**: 7

(i) Write the dual of the LPP:

Maximize $Z = 3x_1 + x_2 + x_3 - x_4$

Subject to the constraints

$$x_1 + 5x_2 + 3x_3 + 4x_4 \leq 4$$
$$x_1 + x_2 = -1$$
$$x_3 - x_4 \leq -5$$
$$\text{and } x_1, x_2, x_3, x_4 \geq 0$$

(ii) Solve the following LPP:

Maximize $z = 2x_1 + 3x_2$

subject to: $x_1 + 3x_2 \leq 5$

$5x_1 + 4x_2 = 12$

and $x_1, x_2 \geq 0$.

(b) Attempt any **ONE**: 4

(i) State the advantages and limitations of the Linear Programming Model

(ii) State the basic results of duality

(c) Attempt any **ONE**: 3

(i) Define Basic Solution and Basic Feasible Solution

(ii) Discuss the role of slack, surplus and artificial variables

2 (a) Attempt any **ONE**: 7

(i) Find the optimum integer solution to the LPP

Maximize $Z = x_1 + x_2$

Subject to the constraints : $x_1 + 2x_2 \leq 4$

$2x_1 + 7x_2 \leq 5$

$x_1, x_2 \geq 0$

(ii) Use Gomory's cutting plane method to solve the LPP

Maximize $Z = 3x_1 + 2x_2$

Subject to the constraints : $2x_1 + 3x_2 \leq 6$

$5x_1 \leq 25$

$x_1, x_2 \geq 0$ and integers.

P.T.O.

E759-2

(b) Attempt any **ONE**: 4

(i) What is the need for integer programming problems?

(ii) Discuss steps of Gomory's cutting plane method.

(c) Attempt any **ONE**: 3

(i) What is the role of cutting plane constraint in integer programming problem

(ii) Discuss the Branching and Bounding terms

3(a) Attempt any **ONE**: 7

(i) Find the optimal cost of transportation using VAM

	P	Q	R	Supply
A	26	23	10	61
B	14	13	21	49
C	16	17	29	90
Demand	52	68	80	

(ii) Find the optimal assignment schedule

	I	II	III	IV	V
A	10	12	15	12	8
B	7	16	14	14	11
C	13	14	7	9	9
D	12	10	11	13	10
E	8	13	15	11	15

(b) Attempt any **ONE**: 4

(i) Discuss the nature of the solution based on opportunity costs

(ii) Discuss the steps to be performed in solving assignment problem with objective of maximization?

(c) Attempt any **ONE**: 3

(i) Why one cannot use simplex method in solving transportation problem?

(ii) How to resolve degeneracy in a transportation problem

4(a) Attempt any **ONE**: 7

(i) Compute auxiliary equations (only) to

Maximize $Z = 2x_1x_2 - x_2^2 + 3x_1$

subject to the constraints : $x_1 + 2x_2 \le 4,\ 2x_1 + x_2 \le 2;\ x_1, x_2 \ge 0$.

E789-3

(ii) Use Beale's Method to obtain the solution of following non-linear programming problem

Maximize $Z = 4x_1 - x_1^2 + 3x_2$ subject to the constraints $3x_1 + x_2 \leq 3$

(b) Attempt any **ONE**: 4

(i) How is a LPP different from NLPP?

(ii) Define quadratic form and comment upon the nature of the objective function

(c) Attempt any **ONE**: 3

(i) What are Kuhn-Tucker's necessary and sufficient conditions?

(ii) Discuss steps of Beale's method.

5(a) Attempt any **ONE**: 7

(i) Using dynamic programming, Maximize $z = y_1^2 + y_2^2$ subject to $y_1 + y_2 = 6$, $y_1, y_2 \geq 0$.

(ii) Maximize $\dfrac{x_1 + x_2 + 5}{2x_1 + 3x_2 + 6}$

subject to: $x_1 + 2x_2 \leq 2$, $3x_1 + 7x_2 \leq 21$; $x_1, x_2 \geq 0$

(b) Attempt any **ONE**: 4

(i) What are the characteristics of dynamic programming?

(ii) Discus about the components of Dynamic Programming Problem

(c) Attempt any **ONE**: 3

(i) When will a fractional programming problem have a solution?

(ii) Define (i) Dynamic programming, (ii) Recursive function