0305E327 Candidate's Seat No : ____________

M.Sc. (Sem.-II) Examination

407

Mathematics

May-2017

Time : 3 Hours] [Max. Marks : 70

1 (A). A cycloid is parametrized as [7]

$$\gamma(t) = (t - \sin(t), 1 - \cos(t)).$$

Calculate the arclength of the cycloid between the points $(0, 0)$ and $(2\pi, 0)$.

OR

1 (A). Define a regular curve. [7]

Which of the following curves are regular?

(i) $\gamma(t) = (\cos^2(t), \sin^2(t)), \quad -\infty < t < \infty,$

(ii) $\gamma(t) = (\sin^2(t), \cos^2(t)), \quad 0 < t < \frac{\pi}{2},$

(iii) $\gamma(t) = (t, \cosh(t)), \quad -\infty < t < \infty.$

1 (B). Answer any two. [4]

(i) Find a parametrization for the curve
$$y^2 - x^2 = 1.$$

(ii) Find a parametrization for the curve
$$2y + 3x = 2.$$

(iii) Find a cartesian equation for the curve
$$\gamma(t) = (t^2 + 1, 2t).$$

1 (C). Answer all. [3]

(i) Find the tangent vector to the curve $\gamma(t) = (\cos(t), \sin(t))$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(ii) Find the tangent vector to the curve $\gamma(t) = (t, \log(t))$ at the point $(e, 1)$.

(iii) Find the tangent vector to the curve $\gamma(t) = (t, t^2, t^3)$ at the point $(2, 4, 8)$. (PTO)

2 (A). Compute $k, \tau, \bar{t}, \bar{n}, \bar{b}$ for the curve [7]

$$\gamma(t) = \left(\tfrac{1}{3}(1-t)^{3/2}, \tfrac{1}{3}(1+t)^{3/2}, \tfrac{-t}{\sqrt{2}}\right).$$

Verify that the Frenet-Serret equations are satisfied.

OR

2(A). Define the torsion of a unit speed curve $\gamma(s)$, whose curvature vanishes nowhere. [7]

Suppose $\Gamma(t)$ is a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Give a formula for the torsion of Γ in terms of Γ and its derivatives. (Do not prove).

Compute the torsion of the circular helix

$$\gamma(\theta) = (a\cos(\theta), a\sin(\theta), b\theta).$$

2 (B) Answer any two. [4]

(i) Is the curve $\gamma(t) = (1+t, 1-t, 3t+2)$ a planar curve?

(ii) Find the signed curvature of the curve

$$\gamma(t) = \left(\tfrac{1}{2}\cos(2t) - 1, 3 + \tfrac{1}{2}\sin(2t)\right)$$

at the point $(-1, \frac{7}{2})$.

(iii) If γ is a unit-speed plane curve show that $\dot{n}_s = -k_s t$.

2 (C) Answer all. [3]

(i) Suppose $\gamma(s)$ and $\Gamma(s)$ are two unit-speed curves in $\mathbb{R}^3$ with the same curvature $k(s) > 0$ and the same torsion $\tau(s)$ for all s. How are the curves related?

2 (c).

(ii) Write down a formula (without proof) for the curvature of a regular curve $\gamma(t)$ in $\mathbb{R}^3$.

(iii) Write down the Frenet-Serret equations for a unit speed curve $\gamma(s)$ in $\mathbb{R}^3$, (Do not prove).

3 (A). Let $f(x, y) = x^2 + y^2 - 2x$. [7]

Show that the graph of $f(x, y)$,

$$S = \{ (x, y, z) \in \mathbb{R}^3 \mid z = f(x,y) \},$$

is a smooth surface with atlas consisting of a single regular surface patch

$$\sigma(u, v) = (u, v, f(u, v)).$$

OR

3(A). Show that the level surface [7]

$$\frac{x^2}{3^2} - \frac{y^2}{2^2} - \frac{z^2}{5^2} = 1$$

is a smooth surface.

3(B). Answer any two. [4]

(i) Find the equation of the tangent plane to the surface patch $\sigma(u, v) = (u, v, u^2 - v^2)$ at the point $(3, 1, 8)$.

(ii) Give a tangent vector to the surface $2x^2 - 3x + 2y + z = 0$, at the point $(1, 1, -1)$.

(iii) Show that the unit sphere cannot be covered by a single surface patch. (P.T.O)

3(C). Answer all. [3]

(i) Give a parametrization (without proof) of the plane $2x + 3y + 4z + 5 = 0$.

(ii) Give a parametrization (without proof) of the unit sphere $x^2 + y^2 + z^2 = 1$.

(iii) Give a parametrization (without proof) of the surface $z = \sin(x) + \cos(y)$.

4(A). Show that [7]

$$\sigma(u, v) = (\operatorname{sech}(u)\cos(v), \operatorname{sech}(u)\sin(v), \tanh(u))$$

is a regular surface patch for the unit sphere. Show that the meridians and parallels on the sphere correspond under σ to perpendicular straight lines in the plane.

OR

4(A). A quadric is defined by [7]

$$y^2 + 2z^2 + 3x + 6y - 4z = 7,$$

Describe the quadric and give a parametrization.

4(B). Answer any two. [4]

(i) Define a generalised cylinder and give a parametrization $\sigma(u, v)$.

(ii) Define a generalised cone and give a parametrization $\sigma(u, v)$.

(iii) Define a ruled surface and give a parametrization $\sigma(u, v)$.

4 (c). Answer all. [3]

(i) Draw a hyperboloid of one sheet.

(ii) Give an example (without proof) of a triply orthogonal system of surfaces.

(iii) Give an example (do not prove) of a quadric surface which is a ruled surface.

5 (A). Define a conformal map. Show that (Mercator's) parametrization of the sphere [7]

$$\sigma(u, v) = (\operatorname{sech}(u)\cos(v), \operatorname{sech}(u)\sin(v), \tanh(u))$$

is conformal.

OR

5 (A). Define an isometry. Show that a generalised cylinder is isometric to (part of) the plane. [7]

5 (B). Answer any two. [4]

(i) Calculate the first fundamental form of the surface patch

$$\sigma(u, v) = (u - v, u + v, u^2 + v^2).$$

(ii) Give an example of a conformal map that is not an isometry.

(iii) Define an equiareal map. Give a map (other than the identity) from the plane to itself which is equiareal. (P.T.O.)

5 (c). Answer all. [3]

(i) Is the map from the plane $x+y+z=0$ to the xy-plane given by
$(x, y, z) \longmapsto (x, y, 0)$ an isometry ?

(ii) What is the surface area of the unit sphere ? (Do not prove).

(iii) Give a formula (without proof) for the surface area of a torus.

0505E376

Candidate's Seat No :__________

M.Sc. (Sem.-II) Examination

408

Mathematics

May-2017

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any ONE. **7**

   (i) State and prove Cayley's theorem.

   (ii) For every positive integer n, prove that Aut($\mathbb{Z}_n$) is isomorphic to U(n).

   (b) Attempt any TWO. **4**

   (i) For $n \geq 3$, prove that each $\alpha \in A_n$ can be expressed as a product of 3-cycles.

   (ii) Prove or disprove that U(24) is isomorphic to U(20).

   (iii) Let G be the group of non-zero complex numbers under multiplication and let $H = \{z \in G / |z| = 1\}$. Give a geometric description of the cosets of H.

   (c) Answer very briefly. **3**

   (i) Express the m-cycle (1, 2, 3,..,m) as the product of transpositions.

   (ii) If G is a group with $|G|=71$, prove that G is cyclic.

   (iii) If H and K are subgroups of G such that $|H|=24$ and $|K|=65$, find $|H \cap K|$.

2. (a) Attempt any ONE. **7**

   (i) State and prove G/Z theorem.

   (ii) Let G be a finite Abelian group and let p be a prime that divides the order of G, prove that G has an element of order p.

   (b) Attempt any TWO. **4**

   (i) What is the largest order of any element in $U(900)$?

   (ii) How many elements of order 2 are in $Z_{2000000} \oplus \mathbb{Z}_{4000000}$? Explain.

   (iii) Find a subgroup of $\mathbb{Z}_{12} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{15}$ that is of order 9.

(P.T.O)

(c) Answer very briefly. **3**

(i) What is the largest order of any element in $\mathbb{Z}_{27} + \mathbb{Z}_{72}$? Explain.

(ii) What is the order of element $14+ < 6 >$ in the factor group $\mathbb{Z}_{18}/ < 6 >$?

(iii) In $\mathbb{Z}$, let $H =< 5 >$ and $K =< 7 >$, prove $\mathbb{Z} = HK$. Does $\mathbb{Z} = H \times K$? Explain.

3. (a) Attempt any ONE. **7**

(i) Prove that $H_1 \times H_2 \times ... \times H_n$ is isomorphic to $H_1 \oplus H_2 \oplus ... \oplus H_n$.

(ii) State and prove the First Isomorphism Theorem.

(b) Attempt any TWO. **4**

(i) Determine all Abelian groups of order 200.

(ii) If G is a finite group and the factor group G/H has an element of order k then prove that G has an element of order k.

(iii) Can there be a homomorphism from $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ onto $\mathbb{Z}_8$? Justify.

(c) Answer very briefly. **3**

(i) Prove that SL(2, $\mathbb{R}$) is a normal subgroup of GL(2, $\mathbb{R}$).

(ii) Prove or disprove: If H and K are subgroups of group G then HK is also a subgroup of G.

(iii) Determine all homomorphisms from ($\mathbb{Z}$,+) to itself.

4. (a) Attempt any ONE. **7**

(i) Let G be a finite group such that $|G| = p^k$ (where, p is a prime; $k \geq 1$)then prove that $Z(G)$ has more than one element.

(ii) State and prove Sylow's First Theorem.

(b) Attempt any TWO. **4**

(i) If $|G| = 36$ and G is non-Abelian then show that $n_2 > 1$ or $n_3 > 1$.

(ii) Prove that any group of order 99 is Abelian.

(iii) If H is a Sylow p-subgroup of a finite group G and H is normal in G then show that G has a unique Sylow p-subgroup.

(c) Answer very briefly. E376-3 3

(i) Define the conjugacy class $cl(a)$.

(ii) State (without proof) the Sylow's third theorem

(iii) Prove that every group of order p^2 (where, p is prime) is Abelian.

5. (a) Attempt any ONE. 7

(i) Define simple group. Prove that there are no simple groups of order 80 and 112.

(ii) Define simple group. Prove that A_5 is simple.

(b) Attempt any TWO. 4

(i) Determine all finite Abelian groups that are simple.

(ii) Prove that there is no simple group of order 280.

(iii) How many non-equivalent ways are there to color three black and three white vertices of a regular hexagon under the subgroup H of all rotations in dihedral group D_6? Explain.

(c) Answer very briefly. 3

(i) If $|G| = p$ (p is prime), show that G cannot be simple.

(ii) Define (a) orbit(i) (b) $fix(\phi)$.

(iii) State (without proof) Burnside theorem.

1005E467 Candidate's Seat No : ________

M.Sc. (Sem.-II) Examination

410

Mathematics

May-2017

Time : 3 Hours] [Max. Marks : 70

Q.1

A) Attempt any **ONE**:

(i) Find the general integral of $(mz - ny)p + (nx - lz)q = ly - mx$

(ii) Verify that the Pfaffian differential equation $yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0$ is integrable and find its integral.

B) Attempt any **TWO**:

(i) Eliminate the function F from $z = x^nF(\frac{y}{x})$ and find the corresponding partial differential equation

(ii) If $\vec{X}.curl\vec{X} = 0$ where $\vec{X} = (P, Q, R)$ and μ is an arbitrary differentiable function of x,y and z, then prove that $\mu\vec{X}.curl(\mu\vec{X}) = 0$

(iii) Find the integral of the Pfaffian differential equation $ydx + xdy + 2zdz = 0$

C) Answer very briefly:

(i) In case of first order partial differential equations define a semi-linear equation and give an example

(ii) Give geometrical interpretation of the complete integral of a partial differential equation of the first order

(iii) Find the general integral of $z(xp + yq) = xy$

Q.2

A) Attempt any **ONE**:

(i) Find a complete integral of $p_1^3 + p_2^2 + p_3 = 1$ by Jacobi's method

(ii) Find the integral surface of the linear partial differential equation $xp + yq = z$ which contains the circle defined by $x^2 + y^2 + z^2 = 4, x + y + z = 2$

B) Attempt any **TWO**:

(i) Find a complete integral of $x(1 + y)p = y(1 + x)q$

(ii) Find a complete integral of $pq = px + qy$

(iii) Write (do not solve) the auxiliary equations of the partial differential equation $z^2 = pqxy$

(P.T.O)

C) Answer very briefly:

(i) Write down the two conditions under which the equations $f(x,y,z,p,q) = 0$ and $g(x,y,z,p,q) = 0$ are compatible on a domain D.

(ii) Find the singular integral of $p^2 + q^2 = m^2$, where *m* is a constant

(iii) Define: Admissible Curve

Q.3

A) Attempt any **ONE**:

(i) State and solve the one-dimensional wave equation in case of an infinite vibrating string with initial displacement distribution *f(x)* and initial velocity distribution *g(x)*

(ii) Find the integral surface of the equation $pq=z$ passing through the curve C: $x=0$, $y^2=z$

B) Attempt any **TWO**:

(i) For the one parameter family of planes $z - z_0 = p(x - x_0) + q(y - y_0)$, where $q=q(x_0,y_0,z_0,p)$, write down the analytic expression for the Monge cone at (x_0,y_0,z_0)

(ii) State the solution of the PDE: $u_{tt} = 9u_{xx}$, $-\infty < x,t < \infty$ and initial conditions $u(x,0) = 0, u_t(x,0) = sinx$

(iii) For what values of x and y are the two equations $u_{xx} + 2xu_{xy} + (1 - y^2)u_{yy} = 0$ and $(1 + y^2)u_{xx} + (1 + x^2)u_{yy} = 0$ hyperbolic?

C) Answer very briefly:

(i) Write the expression of the homogeneous second order elliptic partial differential equation in polar coordinates

(ii) Define: Robin's boundary value problem

(iii) State the Cauchy problem for a second order partial differential equation

Q.4

A) Attempt any **ONE**:

(i) State the problem of the wave equation in the case of vibrations of a string of finite length and solve it using the method of separation of variables. Consider that both ends are fixed and initial displacement distribution if *f(x)* and initial velocity distribution is *g(x)*

(ii) State and prove the maximum principle for a two dimensional harmonic function. Also state (only) the strong maximum principle.

B) Attempt any **TWO**:

(i) State the Hadamard's conditions for a well-posed problem

(ii) In case of Laplace equations, prove that the solution to the Dirichlet problem is stable

(iii) In case of Laplace equations, prove that the solution of the Neumann problem is unique upto the addition of a constant

C) Answer very briefly:

(i) Define: Stable solution

(ii) State (only) Harnack's theorem

(iii) State (only) the Neumann problem for a circle

Q.5

A) Attempt any **ONE**

(i) State and solve the heat conduction problem for an infinite rod case with initial temperature distribution in the rod at time *t=0* given by *f(x)*

(ii) Show that $(x - C)^2 + y^2 = \mu[(x + C)^2 + y^2]$ represents a family of equipotential surfaces and find the corresponding potential function

B) Attempt any **TWO**

(i) Using the Duhamel's principle, state the solution of the wave equation in case of vibrating string of infinite length with initial displacement distribution *f(x)* and initial velocity distribution *g(x)* and the forcing term *h(x,t)*

(ii) State (only) the Dirichlet problem for a circle and state its solution

(iii) State (only) the partial differential equation and boundary conditions for the problem : Find the steady state temperature distribution in a semi-circular plate of radius *a*, insulated on both the faces with a curved boundary kept at a constant temperature U_0 and its boundary diameter kept at zero temperature

C) Answer very briefly

(i) Write (only) the Poisson's equation in two variables

(ii) Define: Family of equipotential surfaces

(iii) What is the necessary condition for the existence of the solution *U* of the problem $\nabla^2 U = 0$ in a bounded domain D, and $\frac{\partial U}{\partial n} = f(s)$ on the boundary B, where $\frac{\partial}{\partial n}$ is the directional derivative along the outward normal.

1205E499 Candidate's Seat No :_________

M.Sc. (Sem.-II) Examination
411 Mathematics
May-2017

Time : 3 Hours] **[Max. Marks : 70**

Q.1. (A) Attempt any one [7]

(1) If the sequence of measurable functions $f_n(x)$ converges to $f(x)$ almost everywhere on a bounded measurable set E, then prove that $f_n \Rightarrow f$.

(2) Let $f : E = [a, b] \to \mathbf{R}$ be measurable. Prove that for given $\sigma > 0$ and $\epsilon > 0$, there exists a continuous function $\psi(x)$ on $[a, b]$ such that $mE(|f - \psi| \geq \sigma) < \epsilon$.

(B) Attempt any two [4]

(1) Let $f : E = [a, b] \to \mathbf{R}$ be measurable. Prove that for given $\epsilon > 0$, there exists a bounded measurable function $g(x)$ on $[a, b]$ such that $mE(f \neq g) < \epsilon$.

(2) If $f_n \Rightarrow f$ and $g_n \Rightarrow g$ then show that $3f_n + g_n \Rightarrow 3f + g$.

(3) Verify Egorov's theorem for the sequence $f_n : (0, 1) \to \mathbf{R}$ defined by $f_n(x) = \frac{1}{1+nx}$.

(C) Answer in brief [3]

(1) State (only) Egorov's theorem.

(2) State (only) Luzin's theorem.

(3) If E denotes the set of rationals in $[0, 1]$, then prove that every real-valued function defined on E is measurable.

Q.2. (A) Attempt any one [7]

(1) Define Bernstein polynomial. If $f(x)$ is a continuous function on $[0, 1]$ then prove that the sequence of its Bernstein polynomials converges uniformly to f on $[0, 1]$.

(2) Prove that $L_p[a, b]$ is complete where $p > 1$.

(B) Attempt any two [4]

(1) Show that $\cos^k x$ is an even trigonometric polynomial for every positive integer k.

(2) Show that convergence in mean implies convergence in measure.

(3) Show that the set of all bounded measurable functions on $[a, b]$ is dense in $L_p[a, b]$ for all $1 \leq p < \infty$.

(C) Answer in brief [3]

(1) State (only) Weierstrass theorem for 2π-periodic continuous functions.

(2) Explain the meaning of $f_n \to f$ weakly in $L_p[a, b]$ where $p > 1$.

(3) True or False: $L_p[a, b] \subset L_1[a, b]$ for all $p > 1$.

(P.T.O)

E499-2

Q.3. (A) Attempt any one [7]

(1) If $f : [a,b] \to R$ is increasing then show that its derivative $f'(x)$ is measurable and

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

(2) Deriving all the necessary results, show that the set of discontinuity of an increasing function is atmost countable.

(B) Attempt any two [4]

(1) For the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

compute its any two derived numbers at the origin.

(2) Let $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1. \end{cases}$
Determine the total variation of f on $[0,1]$.

(3) If f is of finite variation on R, then show that

$$\lim_{x\to\infty} V_x^{\infty}(f) = 0.$$

(C) Answer in brief [3]

(1) Determine the total variation of $f(x) = x^2$ on $[-1,1]$.

(2) Let $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ 2x & \text{if } 1 \leq x \leq 2. \end{cases}$
What is the saltus of f at the point $x = 1$?

(3) Prove or disprove: The function $f(x) = x$ is of bounded variation on $(-\infty,\infty)$.

Q.4. (A) Attempt any one [7]

(1) If $f : [a,b] \to R$ is such that $f'(x)$ is finite everywhere and summable on $[a,b]$, then prove that

$$f(c) = f(a) + \int_a^c f'(t)dt, \quad a < c \leq b.$$

(2) If $\phi : [a,b] \to R$ is such that at every point of $[a,b]$ all the derived numbers of ϕ are non-negative, then prove that ϕ is increasing.

(B) Attempt any two [4]

(1) Prove that every absolutely continuous function is of finite variation.

(2) If $f : [a,b] \to [c,d]$ is absolutely continuous and $g : [c,d] \to R$ is Lipschitz continuous then show that $g \circ f$ is absolutely continuous on $[a,b]$.

E499-3

(3) If f is summable on $[a,b]$, then prove that

$$\phi(x) = \int_a^x f(t)dt$$

is absolutely continuous on $[a,b]$.

(C) Answer in brief **[3]**

(1) Let $\phi(x) = \int_a^x f(t)dt$. If the point $x = u$ is the Lebesgue point of f, then show that $\phi'(u) = f(u)$.

(2) True or False: Every continuous function which is of bounded variation on $[a,b]$ is absolutely continuous.

(3) Is $f(x) = \sqrt{x} + 2x$ absolutely continuous on $[0,1]$? Why?

Q.5. (A) Attempt any one **[7]**

(1) State and prove Riemann-Lebesgue lemma and use it to prove that if $\phi \in L[0,\pi]$ then

$$\lim_{n\to\infty} \int_0^\pi \phi(t)\sin(n+\frac{1}{2})tdt = 0.$$

(2) For $f \in L_2[-\pi,\pi]$, if $S_N(x)$ denotes the partial sums of the Fourier series of f, then show that $\|f - T_N\|_2 \geq \|f - S_N\|_2$, for every trigonometric polynomial T_N of degree N.

(B) Attempt any two **[4]**

(1) Show that the series $\sum_{k=0}^{\infty}(-1)^k$ is cesaro-summable.

(2) State (only) Parseval's identity and use it prove that if the Fourier coefficients of an L_2-function are all zero, then the function is zero almost everywhere.

(3) Give the definition of $D_n(x)$ and if x is not a multiple of 2π, then derive the formula for $D_n(x)$ in terms of sine function.

(C) Answer in brief **[3]**

(1) Determine the value of $F_n(x)$ when x is a multiple of 2π.

(2) State any one difference between the properties of $F_n(x)$ and $D_n(x)$.

(3) State (only) any one sufficient condition for the pointwise convergence of the Fourier series for $f \in L[-\pi,\pi]$.