0805E426 Candidate's Seat No :__________

M.Sc. (Sem.-II) Examination

409

Mathematics

May-2017

Time : 3 Hours] **[Max. Marks : 70**

MAT409:COMPLEX ANALYSIS-II

1. (a) If f is analytic on an open disk $|z - z_0| < R_0$, show that $f(z)$ has the series representation: (7)

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad (|z - z_0| < R_0)$$

OR

(a) Show that the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function $S(z)$ at each point inside the circle of convergence $|z - z_0| = R$. (7)

(b) Answer any **two** of the following briefly: (4)

(i) Obtain $\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \cdot \frac{1}{z} + \frac{7}{360}z + \ldots 0 < |z| < \pi$.

(ii) Represent the function $f(z) = \frac{z+1}{z-1}$ by one of its Laurent Series specifying the domain.

(iii) Represent the function $f(z) = \frac{1}{z(1+z^2)}$ by one of its Laurent Series specifying the domain.

(c) Answer all of the following very briefly: (3)

(i) Obtain the Taylor series for e^z in powers of $z - 1$.

(ii) Obtain the Maclaurin series for the function $\cos z$.

(iii) Show that $\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \quad |z - i| < \sqrt{2}$.

2. (a) Evaluate the following:(i) $\int\limits_{|z|=2} \frac{\cosh \pi z}{z(z^2+1)} dz$ (ii) $\int\limits_{|z|=3} \frac{z^3 e^{1/z}}{1+z^3} dz$. (7)

OR

(a) Suppose C_N denotes the positively oriented boundary of the square whose edges lie along the lines $x = \pm\left(N + \frac{1}{2}\right)\pi$ and $y = \pm\left(N + \frac{1}{2}\right)\pi$, where N is a positive integer. Show that (7)

$$\int\limits_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

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(b) Answer any **two** of the following briefly: (4)

(i) Describe three types of isolated singular points with an illustration of each type.

(ii) Evaluate the integral: $\int_{|z|=3} \frac{\exp(-z)}{(z-1)^2} dz$.

(iii) Evaluate the integral $\int_{|z|=2} \frac{1}{1+z^2} dz$.

(c) Answer all of the following very briefly: (3)

(i) List all the singular points of $\frac{1}{\sin(\frac{\pi}{z})}$. Which of these are isolated singular points and which are not?

(ii) List all the singular points of $\frac{z+1}{z^3(z^2+1)}$. Which of these are isolated singular points and which are not?

(iii) Find $\operatorname{Res}_{z=i} \frac{1}{(z-i)^2}$.

3. (a) State and prove Liouville's theorem. What does this theorem say for the entire function that is not a constant function? What can you conclude for the function $\exp(z)$ or $\sin(z)$? (7)

OR

(a) Suppose $f(z)$ is analytic and $|f(z)| \leq |f(z_0)|$ on $|z - z_0| < \epsilon$. Show that f is constant throughout the neighbourhood. (7)

(b) Answer any **two** of the following briefly: (4)

(i) State the Maximum Modulus Principle. And derive the Minimum Modulus Principe after carefully stating it.

(ii) Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound; that is $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

(iii) Suppose $f(z) = e^z$ and R is the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$. Where does $u(x, y) = Re[f(z)]$ reach its maximum and minimum on R?

(c) Answer all of the following very briefly: (3)

(i) State carefully the Fundamental theorem of algebra.

(ii) Let $f(z) = (z + 1)^2$ and R be the closed triangular region determined by 0, 2 and i. Where do the maximum and minimum of

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$|f(z)|$ occur on R.

(iii) $|f(z)|$ can have its minimum value at an interior point of R. Justify the statement.

4. (a) Giving all the details evaluate the improper integral $\int_0^\infty \frac{x^2}{x^6+1}dx$ using residues. (7)

OR

(a) Giving all the details evaluate the improper integral $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}dx$ using residues. (7)

(b) Answer any <u>two</u> of the following briefly: (4)

(i) Giving minimal details show that $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi$

(ii) Giving the main steps only and using residues find the value of

$$\int_0^\infty \frac{1}{x^3+1}dx$$

(iii) Under the appropriate assumptions show that:

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

What is this result called?

(c) Answer all of the following very briefly: (3)

(i) How in two ways can one define the Improper Integral $\int_{-\infty}^{\infty} f(x)dx$?

(ii) Using residues and giving minimal details find the value of:

$$\int_0^\infty \frac{1}{x^2+1}dx$$

(iii) How do you convert $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$ to the contour integral?

5. (a) Suppose f is meromorphic in the domain interior to a positively oriented simple closed contour C, and f is analytic and nonzero on C. Then show that the winding number of $\Gamma = f(C)$ around origin is given by (7)

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P$$

What are Z and P?

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OR

(a) Give carefully the definition of Möbius Transformation as a bijection from the extended complex plane onto the extended complex plane. Show that composition of two Möbius Transformations is also a Möbius Transformation. Show that the inverse of a Möbius Transformation is also a Möbius Transformation. (7)

(b) Answer any **two** of the following briefly: (4)

(i) Show that every linear fractional transformation, with one exception, has at most two fixed points in the extended complex plane. State clearly as to what is this exception?

(ii) Show that any bilinear transformation is completely determined by its effect on any three distinct points.

(iii) Counting multiplicities determine the number of roots of the polynomial equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \leq |z| < 2$.

(c) Answer all of the following very briefly: (3)

(i) With the use of indented path how will you obtain $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ narrate the story without elaborating too much.

(ii) Find all the fixed points of $Tz = \frac{z-1}{z+1}$ in the extended complex plane.

(iii) Give an example of Möbius Transformation which has exactly one fixed point in the extended complex plane. Which is the fixed point of your example?