Seat No. : _______________

XY-111

April-2013

M.Sc. (Sem.-II)

Mathematics : 408

(Algebra – I)

Time : 3 Hours] **[Max. Marks : 70**

Instruction: (1) There are **five** questions.

(2) Each question carry equal **14** marks.

1. (a) Prove that disjoint cycle commute. **7**

OR

Prove that for any n, U(n) is isomorphic to Aut($\mathbb{Z}_n$).

(b) Attempt any **two** : **4**

(1) Prove that (Q, +) is not isomorphic to (R^+, $*$).

(2) Let G be a group. Prove or disprove : H = $\{g^2 / g \in G\}$ is a subgroup of G.

(3) Suppose G is an Abelian group with order (2n + 1), show that the product of all the elements of G is the identity.

(c) Attempt **all** : **3**

(1) How many elements of order 2 are there in A_5 ?

(2) If $\alpha, \beta \in S_n$ prove that $\alpha^{-1}, \beta^{-1}\alpha\beta$ is even.

(3) Prove or disprove : U(8) is isomorphic to U(12).

2. (a) Let G and H be finite cyclic groups. Then prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. **7**

OR

Let G be a finite Abelian group and let p be a prime that divides $|G|$. Prove that G has an element of order p.

(b) Attempt any **two** : **4**

(1) Prove or disprove : $\mathbb{Z} \oplus \mathbb{Z}$ is cyclic.

(2) Let G = $\{3^m 6^n \mid m, n \in \mathbb{Z}\}$ under multiplication prove that G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

(3) Let G = U(32) and K = {1, 15}, prove that G/K is isomorphic to $\mathbb{Z}_8$.

(c) Attempt **all** : **3**

(1) Find the last two digits of 23^{123}.

(2) What is the largest order of any element in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$?

(3) Prove that the centre Z(G) of group G is always a normal subgroup of G.

3. (a) Define Internal direct product of H and K. Express U(105) as a direct product of two subgroups H, K. **7**

OR

Let ϕ be a homomorphism from a group to a group $\bar{G}$ and let $g \in G$. If $|g| = n$ then prove that $|\phi(g)|$ divides n.

(b) Attempt any **two** : **4**

(1) Determine all homeomorphisms from $\mathbb{Z}_n$ to itself.

(2) What is the smallest positive integer n such that there are exactly 4 non-isomorphic Abelian groups of order n.

(3) Find all Abelian groups (upto isomorphism) of order 360.

(c) Attempt **all** : **3**

(1) Let $G = D_n$. Find a homomorphism from D_n to the multiplicative group $\{1, -1\}$.

(2) Define f from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ by $f(n) = 3n$. Is f a homomorphism ? Explain.

(3) Determine the isomorphism class of U(12).

4. (a) State and prove Sylow's first theorem. **7**

OR

If G is a group of order pq, where p and q are primes, $p < q$ and p does not divide $(q - 1)$ then prove that G is cyclic.

(b) Attempt any **two** : **4**

(1) Determine the groups of order qq.

(2) If a group G has only one p-Sylow subgroup, prove that the subgroup is normal.

(3) Prove that a non-cyclic group of order 21 must have 14 elements of order 3.

(c) Attempt **all** : **3**

(1) What is the smallest possible odd integer that can be the order of a non-Abelian group ?

(2) Prove that conjugacy is an equivalence relation.

(3) How many 3-Sylow subgroups does S_5 have ?

5. (a) Let n be a positive integer that is not prime, and let p be a prime divisor of n. If 1 is the only divisor of n that is congruent to 1 modulo p, then prove that there does not exist a simple group of order n. **7**

OR

Define simple group. Prove that A_5 is simple.

(b) Attempt any **two** : **4**

(1) If $|G| = 112$, prove that G cannot be simple.

(2) Show that for $n \geq 3$, S_n is not simple.

(3) State (without proof) Index theorem and Embedding theorem.

(c) Attempt **all** : **3**

(1) If $|G| = p$, p prime prove that G is simple.

(2) If $|G| = p^2$, p prime, prove that Z(4) is non-trivial.

(3) Does there exists a simple non-Abelian group of order 168 ?