Seat No. : _______________

AX-112

May-2016

M.Sc., Sem.-II

411 : Mathematics
(Real Analysis)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

(1) Prove that convergence in measure need not imply pointwise convergence in general.

(2) State and prove Riesz theorem.

(b) Attempt any **two** : **4**

(1) Verify Egorov's theorem for the sequence $f_n : [0, 1] \to R$ defined by $f_n(x) = x^{n^2}$.

(2) If $f_n \Rightarrow f$ and $g_n \Rightarrow g$ then show that $f_n + 2g_n \Rightarrow f + 2g$.

(3) If $f_n \Rightarrow f$ and g is a bounded measurable function then show that $f_{ng} \Rightarrow fg$.

(c) Answer in brief : **3**

(1) True or False : If $f_n \Rightarrow f$ then $|f_n| \Rightarrow |f|$.

(2) If E denotes the set of rationals in [0,1], then prove that every real-valued function defined on E is measurable.

(3) Define : Convergence in measure.

2. (a) Attempt any **one** : **7**

(1) Define Bernstein polynomial. If $f(x)$ is a continuous function on [0, 1] then prove that the sequence of its Bernstein polynomials converges uniformly to f on [0, 1].

(2) Show that the set of all bounded measurable functions and the set of all continuous functions on [a, b] is dense in $L_p[a, b]$ for all $1 \leq p < \infty$.

(b) Attempt any **two** : **4**

(1) If f, g $\in$ G $L_p[a, b]$, then show that $2f - 3g \in L_p[a, b]$.

(2) Using the theorem of Bernstein polynomials deduce that if $f : [a, b] \to R$ is continuous then for every $\varepsilon > 0$ there exists a polynomial function $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$.

(3) State and prove Minkowski's inequality for functions.

(c) Answer in brief : **3**

(1) How do we define a norm in $L_p[a, b]$?

(2) Express $\cos^2(x + 2)$ in the form of a trigonometric polynomial.

(3) True or False : $L_3[a, b] \subset L_1[a, b]$.

3. (a) Attempt any **one** : **7**

(1) If $f : [a, b] \to R$ is increasing then show that its derivative $f'(x)$ is measurable and

$$\int_a^b f'(x)\, dx \leq f(b) - f(a).$$

(2) If $E \subset [a, b]$ is of measure zero, then show that there exists a continuous increasing function $\sigma(x)$ on $[a, b]$ such that $\sigma'(x) = +\infty$ on E.

(b) Attempt any **two** : **4**

(1) Compute the derived numbers of the function $f(x) = |x|$ at $x = 0$.

(2) Let $f(x) = \begin{cases} x + 2 & \text{if } 0 \leq x < 1 \\ 4x & \text{if } 1 \leq x \leq 2 \end{cases}$

Determine the total variation of f on [0, 2].

(3) If f is of finite variation on R, then show that

$$\lim_{x \to \infty} V_x^{\infty}(f) = 0$$

(c) Answer in brief : **3**

(1) Give the definition of derived number.

(2) Let $f(x) = \begin{cases} x + 2 & \text{if } 0 \leq x < 1 \\ 2x & \text{if } 1 \leq x \leq 2 \end{cases}$

What is the saltus of f at the point $x = 1$?

(3) True or False : Every function of finite variation on [a, b] is bounded.

4. (a) Attempt any **one** : **7**

(1) If f : [a, b] $\to$ R is such that $f'(x)$ is finite everywhere and summable on [a, b], then prove that

$$f(c) = f(a) + \int_a^c f'(t)\, dt,\ a < c \le b.$$

(2) If f : [a, b] $\to$ R is absolutely continuous and $f'(x) = 0$ almost everywhere then prove that $f(x)$ is constant function.

(b) Attempt any **two** : **4**

(1) Prove that every absolutely continuous function is of finite variation.

(2) Prove that the product of two absolutely continuous functions is an absolutely continuous function.

(3) Show that every C^1 function on [a, b] is absolutely continuous.

(c) Answer in brief : **3**

(1) Let $\phi(x) = \int_a^x f(t)\, dt$. If the point $x = u$ is the Lebesgue point of f, then show that $\phi'(u) = f(u)$.

(2) Give an example of a differentiable function f on [0, 1] whose derivative is not Lebesgue integrable on [0, 1].

(3) True or False: Every Lipschitz continuous function on [a, b] is absolutely continuous.

5. (a) Attempt any **one** : **7**

(1) Show that if $f \in L[-\pi, \pi]$ is continuous at the point $x_0 \in (-\pi, \pi)$, then its Fourier series is cesaro summable at the point x_0 to $f(x_0)$.

(2) State and prove Riemann-Lebesgue lemma and use it to prove that if $f \in L[-\pi, \pi]$ is differentiable at the point $x_0 \in (-\pi, \pi)$, then $S_N(x_0) \to f(x_0)$, as $N \to \infty$, where $S_N(x_0)$ denotes the partial sums of the Fourier series of f at the point x_0.

(b) Attempt any **two** : **4**

(1) Define Fejer Kernel $F_N(x)$ and show that $F_N(x) \geq 0$ for all N and all x.

(2) Show that if the series $\Sigma\ c_n$ is cesaro-summable and $c_n \geq 0$ for all n, then $\Sigma\ c_n$ is summable (convergent).

(3) State and prove Bessel's inequality for $f \in L_2[-\pi, \pi]$.

(c) Answer in brief : **3**

(1) Show that

$$\frac{2}{\pi} \int_0^{\pi} D_N(x)dx = 1.$$

(2) True or False : The series $\sum_{n=1}^{\infty} (-1)^{n+1}$ is (C, 1) summable.

(3) Can we say that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}} + \frac{\cos nx}{n}$ is a Fourier series for some function in $L_2[-\pi, \pi]$? Why ?