Seat No. : _______________

AT-129

May-2016

M.Sc., Sem.-II

409 : Mathematics

(Complex Analysis – II)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Suppose that a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$. Then show that it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$. **7**

OR

Show that the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function S(z) at each point inside the circle of convergence $|z - z_0| = R$.

(b) Answer any **two** of the following briefly : **4**

(i) Give Laurent Series Expansions for the function $f(z) = \dfrac{1}{z^2(1 - z)}$ valid in the domains (i) $1 < |z| < \infty$ (ii) $0 < |z - 1| < 1$.

(ii) Obtain Maclaurin series for sinh z.

(iii) Write z^5 in power of $z - 2$.

(c) Answer **all** of the following very briefly : **3**

(i) Show that

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} \frac{(z - i)^n}{(1 - i)^{n+1}} \; |z - i| < \sqrt{2}$$

(ii) Find $\operatorname*{Res}_{z=0} \frac{e^z}{z^2}$.

(iii) Obtain $e^z = e \sum_{n=0}^{\infty} \dfrac{(z - 1)^n}{n!}$.

2. (a) Suppose f is analytic at a point z_0. Show that z_0 is a zero of order m if and only if there is a function g which is analytic and non-zero at z_0 such that $f(z) = (z - z_0)^m g(z)$. Illustrate this result for the function $f(z) = z(e^z - 1)$ finding order of zero by determining the appropriate g with the required properties. **7**

OR

Show that an isolated singular point z_0 of a function f is a pole of order m if and only if f(z) can be written in the form $f(z) = \frac{\phi(z)}{(z - z0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0. Also show in this case that $\underset{z = z_0}{\text{Res}} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

(b) Answer any **two** of the following briefly : **4**

(i) Find the value of the integral $\int_{|z|=2} \frac{z^5}{1 - z^3} dz$ by calculating a single residue.

(ii) Evaluate the integral : $\int_{|z|=1} z^2 \sin\left(\frac{1}{z}\right) dz$.

(iii) Find $\underset{z = i}{\text{Res}} \frac{z^{\frac{1}{2}}}{(z^2 + 1)^2}$ $(|z| > 0, 0 < \arg z < 2\pi)$

(c) Answer all of the following very briefly : **3**

(i) Describe all the singular points of $\frac{1}{\sin\left(\frac{\pi}{z}\right)}$. Which of these are isolated singular points and which are not ?

(ii) Find the order of the pole and residue at origin for the function $f(z) = \frac{1 - \cosh z}{z^3}$.

(iii) Determine the order of any one pole for $f(z) = \frac{\exp z}{z^2 + \pi^2}$ and also find the corresponding residue.

3. (a) Show that every non-constant, entire function is unbounded. **7**

OR

Show that an analytic function f is constant on $\in$-neighbourhood of z_0 if its modulus has a maximum value at z_0.

(b) Answer any **two** of the following briefly : **4**

(i) Show that Minimum Modulus principle is valid if one extra condition is satisfied.

(ii) Let $f(z) = (z + i)^2$ and R be the closed triangular region determined by –1, 1 and i. Relying on geometric argument, determine the points on R where the maximum and minimum of $|f(z)|$ occur.

(iii) Suppose that f(z) is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is $u(x, y) \leq u_0$ for all points (x, y) in the xy-plane. Show that $u(x, y)$ must be constant throughout the plane.

(c) Answer **all** of the following very briefly : **3**

(i) State the fundamental theorem of algebra.

(ii) State the Maximum Modulus Principle.

(iii) Where does the Modulus of $f(z) = z - i$ has its minimum value ? And what is this minimum value ?

4. (a) Giving all the details and using residues evaluate the improper integral

$$\int_0^\infty \frac{x^2}{(x^2 + 9)(x^2 + 4)^2}\, dx.$$ **7**

OR

Giving all the details and using residues evaluate the improper integral

$$\int_0^\infty \frac{dx}{(x^2 + 1)^2}$$

(b) Answer any **two** of the following briefly : **4**

(i) How can one define improper integrals in two different ways ? Under what condition are both the definitions equivalent ? Justify.

(ii) Giving the main steps only and using residues find :

$$\int_0^\infty \frac{dx}{(x^2 + 1)}$$

(iii) Giving the main steps only and using residues find :

$$\text{P.V.} \int_{-\infty}^\infty \frac{x \sin x}{(x^2 + 2x + 2)}\, dx$$

(c) Answer all of the following very briefly : **3**

(i) State Jordan's Lemma

(ii) Find $\underset{z=-i}{\text{Res}} \frac{z^2}{z^6+1}$

(iii) Find $\underset{z=i}{\text{Res}} \frac{e^{i3z}}{(z^2+1)^2}$

5. (a) Suppose f is meromorphic in the domain interior to a positively oriented simple closed contour C, and f is analytic and nonzero on C. Then show that the winding number of $\Gamma = f(C)$ around origin is given by **7**

$$\frac{1}{2\pi} \Delta_C \arg f(z) = Z - P$$

What are Z and P ?

OR

Show that every linear fractional transformation, with one exception, has at most two fixed points in the extended complex plane. State clearly as to what is this exception. Derive from this result that there is only one linear fractional transformation that maps three distinct points z_1, z_2, z_3 of the extended complex plane onto three distinct points w_1, w_2, w_3 of the extended complex plane respectively.

(b) Answer any **two** of the following briefly : **4**

(i) With the use of indented path and giving only main steps obtain

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(ii) Determine the number of roots (counting multiplicities) of the polynomial equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \leq |z| < 2$.

(iii) Find the bilinear transformation from the extended plane to the extended plane that maps the points ∞, i, 0 onto the points 0, i, ∞.

(c) Answer **all** of the following very briefly : **3**

(i) What do you mean by winding number ? Make your definition clearer by a simple illustration.

(ii) Find all the fixed points of $Tz = \frac{z-1}{z+1}$ in the extended complex plane.

(iii) Give an example of Möbius Transformation which has exactly one fixed point in the extended complex plane.