Seat No. : _______________

ND-102

December-2015

B.Sc., Sem.-V

Core Course-303 : Mathematics

(Complex Variables and Fourier Series)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) **All** questions are compulsory.
(2) Q. **5** is of short questions.
(3) Each question carries **14** marks.

1. (a) In the system C of complex numbers define convergence of sequence. **7**

Prove that $\left(\frac{\overline{z_1}}{z_2}\right) = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$, for $z_1, z_2 \in$ C. Obtain the roots of the equation $z^4 - z^3 + z^2 - z + 1 = 0$.

OR

Define trigonometric and hyperbolic functions for the complex variable. Show that $|\sin z|^2 + |\cos z|^2 = \text{ch}^2 y$; $z \in$ C. Also, express $\sqrt{3} - i$ in the exponential form.

(b) State and prove the De Moivre's theorem for the complex numbers. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$. **7**

OR

Define convergence of the series of complex numbers. Suppose that $z_n = x_n + iy_n$ (n = 1, 2, ….) and S = X + iY, then prove that $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

2. (a) Define : Harmonic conjugate of a function, Entire function. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D with non-zero constant modulus, then prove that the function f is constant. **7**

OR

If $f(z) = u(x, y) + iv(x, y)$; $z = x + iy$ and $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, then prove that $\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \lim_{(x,y) \to (x_0 \cdot y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0 \cdot y_0)} v(x, y) = v_0$.

(b) State Cauchy-Riemann equations in the polar form and verify the same for the function z^n. Find the harmonic conjugate of the function $x^3 - 3xy^2 + 2x$ and obtain the corresponding analytic function terms of z. **7**

OR

The function f is defined as $f(z) = \frac{(\bar{z})^2}{z}$; $z \neq 0$ and $f(z) = 0$; $z = 0$, the show that f(z) is not analytic at $z = 0$; even if it satisfies Cauchy-Riemann equations at the origin.

3. (a) Define : Linear transformation, Bilinear transformation and Conformal mapping. Prove : An analytic function f(z) is conformal at z_0 if and only if $f'(z_0) \neq 0$. **7**

OR

If α, β are fixed points of a bilinear transformation $w = f(z)$, then prove that $\frac{w - \alpha}{w - \beta} = \lambda \frac{z - \alpha}{z - \beta}$; where λ is a complex constant. Find the fixed points of $w = \frac{z - 1}{z + 1}$.

(b) Obtain the images of the curves $y = x - 1$and $y = 0$ under the mapping $w = \frac{1}{z}$, $z \neq 0$ also, examine conformality of the mapping at $z = 1$. **7**

OR

Find the critical points of the mapping $w = 2z^3 - 15z^2 + 36z + 7$ and find its angle of rotation at the point $1 - i$. Obtain in image of $| z - i | < 3$ under the mapping $w = \frac{iz + 1}{2i + z}$.

4. (a) Prove : If f(x) is Riemann integrable in $(-\pi, \pi)$, then the series $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges, where a_n and b_n are the Fourier coefficients of f(x). **7**

OR

Define the Fourier series for the function f and obtain the same for the function $f(x) = x \sin x$, hence deduce that $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot3} - \frac{1}{3.5} + \frac{1}{5\cdot7} - \ldots$

(b) Find the Fourier series expansion of the function $f(x) = x - x^2$ in $[-\pi, \pi]$. **7**

OR

Obtain the Fourier series for the function $f(x) = x^2$ in $(0, 2\pi)$.

5. Attempt any **seven** : **14**

(i) For the complex number $z = -1 - i$, find the principal argument Arg (z).

(ii) Find the real and imaginary parts of the function $\frac{\bar{z}}{z}$ where, $z = a + ib \neq 0$.

(iii) State the C-R equations and the derivative of the function $f(z) = u + iv$ in Cartesian form.

(iv) Simplify $\log\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$, also write it in a polar form.

(v) Find the singular points of $\frac{z^2 + 1}{(z-1)(z^2 - 7z + 12)}$ and $|z|^2$.

(vi) Obtain $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$ for all m, n = 0, 1, 2, ….

(vii) Find $\int_{-\pi}^{\pi} \cos^2 nx \, dx$.

(viii) Is the function $f(z) = (\bar{z})^2$ analytic ? Justify.

(ix) State the Bessel's inequality.