Seat No. : _______________

NS–123

December–2015

M.Sc., Sem.–I

403 : Mathematics
(Complex Analysis–I)

Time : 3 Hours] **[Max. Marks : 70**

1\. (a) What do the equations $|z - i| = |z + i|$ and $|z - i| + |z + i| = 2$ represent ? Justify your answers. **7**

OR

How are the n^{th} roots of a complex number $z_0 = r_0 e^{i\theta_0}$ given ? Find all the sixth roots of 8, exhibit them all graphically. Also find fourth roots of –1 and sketch them.

(b) Answer any **two** of the following briefly : **4**

(i) Write $(-1 + i)^7$ in the rectangular form $x + iy$.

(ii) When do you say that point z_1 is closer to the origin than point z_2 ? Which of the points $3 - 2i$ and $1 + 4i$ is closer to origin ? Justify.

(iii) Show that $|\text{Re}z| + |\text{Im}z| \leq \sqrt{2}\,|z|$.

(c) Answer all of the following very briefly : **3**

(i) Sketch the set $|z - 2 + i| = \sqrt{3}$. Is it a domain ?

(ii) Sketch the set $|2z + 3| > 4$ and determine if it is a domain.

(iii) Find $\text{Arg}(-1 - i)$

2\. (a) Suppose $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Show that Cauchy-Riemann equations are satisfied at origin. Does $f'(0)$ exist ? Justify. **7**

OR

Suppose D is a domain and $f : D \to \mathbb{C}$ satisfies $f'(z) = 0$ for all $z \in D$. Show that f(z) is constant on D. By giving an appropriate example show that the condition that D is a domain can not be dropped.

(b) Answer any **two** of the following briefly : **4**

(i) Discuss differentiability of $f(z) = e^{x}e^{-iy}$.

(ii) At which points z of the complex plane $\mathbb{C}$ is the function $f(z) = |z|^2$ differentiable ? Is it analytic anywhere ?

(iii) Show that $\lim_{z \to \infty} f(z) = \infty$ iff $\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$. Using this show that $\lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty$.

(c) Answer all of the following very briefly : **3**

(i) Give one proper subset of $\mathbb{C}$ which is a neighbourhood of ∞.

(ii) When do you say that f is analytic at z_0 ? What do you mean by an entire function ?

(iii) If $f(z) = x^2 + iy^2$, show that $f'(x + ix) = 2x$. Is f analytic anywhere ?

3. (a) When do you say that v is a Harmonic Conjugate of u ? Show that $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a Harmonic Conjugate of u. Also show that if v and V are Harmonic Conjugates of u then they differ by a constant. **7**

OR

How is Log(z) defined on $D = \mathbb{C} - \{0\}$. Discuss the points where it is discontinuous. Justify your claims.

(b) Answer any **two** of the following briefly : **4**

(i) Define $\sin^{-1}z$ giving proper motivation.

(ii) Show that $\sin^{-1}(-i) = n\pi + i(-1)^{n+1}\ln(1 + \sqrt{2})$, $n \in \mathbb{Z}$.

(iii) What is the image of the y–axis under the map $f(z) = \exp(z) = e^{z}$?

(c) Answer all of the following very briefly : **3**

(i) Find the principal value of i^{i}.

(ii) Find the value of Log(– ei).

(iii) Show that $|\exp(z^2)| \le \exp(|z|^2)$.

4. (a) Suppose that f is continuous on a domain D. Show that f has antiderivative F in D if the contour integrals of f(z) around closed contours lying entirely in D all have value zero. **7**

OR

Suppose f is analytic on and within closed region R consisting of all points interior to and on a simple closed contour C and f' is continuous there. Show that

$$\int_C f(z)dz = 0$$

(b) Answer any **two** of the following briefly : **4**

(i) Find the value of : $\int_{i}^{i/2} e^{\pi z}\, dz.$

(ii) Assuming $\int_0^{\pi} e^{(1+i)x}\, dx = \int_0^{\pi} e^{x}\cos x\, dx + i\int_0^{\pi} e^{x} \sin x\, dx$

Evaluate the two integrals on right by evaluating single integral on the left and then comparing the real and imaginary parts.

(iii) Find the integrals $\int_C \frac{1}{z}\, dz$ and $\int_C \bar{z}\, dz$ where C is the right–hand half $z = 2e^{i\theta}\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$.

(c) Answer all of the following very briefly : **3**

(i) What is meant by a simple closed contour ? Explain giving example.

(ii) Giving the meanings of all the notations, establish

$$\left| \int_C f(z)\, dz \right| \le ML$$

(iii) Find the value of $\int_{|z| = 1} e^{\sin z^3} dz.$

5. (a) Stating appropriate assumptions, derive Cauchy Integral Formula. **7**

OR

Stating appropriate assumptions, derive the main part of the proof of the Extension of Cauchy Integral Formula.

(b) Answer any **two** of the following briefly : **4**

(i) Find the value of $\int_C \frac{\cosh (z)}{z^4} dz$, where C is the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(ii) Find $\int_{|z| = 1} \frac{\cos z}{z(z^2 + 8)} dz$

(iii) Find the value of $\int_{|z-i| = 2} \frac{1}{z^2 + 4} dz$.

(c) Answer all of the following very briefly : **3**

(i) State Morera's Theorem.

(ii) Define a Multiply Connected Domain giving an example.

(iii) Find the value of $\int_{|z| = \frac{1}{4}} \frac{z}{2z + 1} dz$.