Seat No. : _______________

NO-112

December-2015

M.Sc., Sem.-I

401 : Mathematics

(Functions of Several Variables)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) **All** questions are compulsory.

(2) Each question carries **14** marks.

1. (a) Attempt any **one** : **7**

(1) Define real valued linear function on E^n. Prove that a real valued function L is linear if and only if there exists real numbers $a_1, a_2, a_3, \ldots, a_n$ such that $L(x) = a.x$ for all $x \in E^n$.

(2) Define convex set. Prove that a set K is convex if and only if every convex combination of points of K is a point of K.

(b) Attempt any **two** : **4**

(1) Define hyperplane. Find the hyperplane in E^4 containing the four points $0, e_1 + e_2, e_1 - e_2 + 2e_3, 3e_4 - e_2$.

(2) Show that if x can be represented in two ways as a convex combination of $x_0, x_1, \ldots\ldots, x_r$, then $x_1 - x_0, x_2 - x_0, \ldots\ldots\ldots, x_r - x_0$ form a linearly dependent set.

(3) Prove that any linear function is continuous.

(c) Attempt **all** : **3**

(1) Prove that any hyperplane is a closed set.

(2) Give two convex subsets of E^3.

(3) Explain the terms : Co-vector, dual of E^n.

2. (a) Attempt any **one** : **7**

(1) Define relative extremum of f at x_0. If f has a relative extremum at x_0 and f is differentiable at x_0, then prove that x_0 is a critical point.

(2) Let f be differentiable on an open, connected domain D such that $df(x) = 0$ for all x in D. Prove that F is constant function on D.

(b) Attempt any **two** : **4**

(1) If f is differentiable at x_0 then prove that f has a derivative at x_0 in every direction ν.

(2) Find the directional derivative of $f(x, y) = xe^{xy}$ at $x_0 = e_1 - e_2$ in the direction $\nu = \frac{1}{\sqrt{2}}(e_1 + e_2)$.

(3) Let $f(x, y) = (x + y + 1)^p$ on $K = \{(x, y) / (x + y + 1) > 0\}$. Determine the values of p for which f is convex on K.

(c) Attempt **all** : **3**

(1) Prove that $f(x) = ax + b$ is a convex function on E.

(2) Define the functions of class $C^{(q)}$. Give an example of a $C^{(3)}$ function that is not $C^{(4)}$.

(3) Is the function $f(x, y) = |xy|$ a differentiable function ? Justify.

3. (a) Attempt any **one** : **7**

(1) Let D be an open subset of E^n and w a continuous 1-form with domain D, then prove that w is exact if and only if for every closed curve lying in D,

$$\int_r w = 0.$$

(2) Define the length of a curve r. If f and g both represent r and f is equivalent to g, then prove that $\int_a^b | f'(\tau) | d\tau = \int_\alpha^\beta |g'(t)|dt$.

(b) Attempt any **two** : **4**

(1) Let r be represented by $f(x) = |x|^{3/2}$ on $[-b, b]$. Find the arc length of r.

(2) Evaluate $\frac{1}{2}\int_r x dy - y dx$, if r is represented by $g(t) = (a \cos t)\, e_1 + (b \sin t)\, e_2$, $0 \le t \le 2\pi$, where $a, b > 0$.

(3) Let D be an open, simply connected subset of E^2 and u and v are functions of class $C^{(1)}$ which satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, then show that for any closed curve lying in D, $\int_r u dx - v dy = 0$ and $\int_r v dx + u dy = 0$.

(c) Attempt **all** : **3**

(1) Find the tangent line at $e_1 + e_2 + e_3$ to the curve represented by

$g(t) = t\, e_1 + t^{1/2}\, e_2 + t^{1/3}\, e_3, \frac{1}{2} \leq t \leq 2.$

(2) When we say that a parametric representation f is equivalent to g ? explain.

(3) Prove that $\int_{-r} w = -\int_r w$.

4. (a) Attempt any **one** : **7**

(1) Define affine transformation. Prove that (r = n) a transformation g is isometry of E^n if and only if g is affine transformation of the form $g(t) = L(t) + x_0$ for all t in E_n and L is orthogonal.

(2) Let r = 3, n = 2 and L be the linear transformation such that $L(\varepsilon_1) = e_1 - 2e_2$, $L(\varepsilon_2) = e_1$, $L(\varepsilon_3) = 5e_1 + e_2$. Find the matrix of L, the rank, and the Kernel.

(b) Attempt any **two** : **4**

(1) State inverse function theorem with a simple example.

(2) Define isometry of E^n. Give two isometries of E^1.

(3) Let $g(s, t) = (s^2 - t^2)\, e_1 + 2st\, e_2$ (n = r = 2). Find the matrix Dg(s, t) and Jacobian Jg(s, t).

(c) Attempt **all** : **3**

(1) If L is a linear transformation from E^r to E^n, prove that the range and the Kernell of L are linear subspaces.

(2) Let $g(s, t) = (s^2 + t^2)\, e_1 + 2st\, e_2$ and $\Delta = E^2$. Draw the region $g(\Delta)$.

(3) Define univalent transformation. Is $g(s, t) = (s^2 + t^2)\, e_1 + (2st) e_2$ univalent ? Justify.

5. (a) Attempt any **one** : **7**

(1) State implicit function theorem with a simple illustration.

(2) Let $\phi(x, y, z) = x^2 + 4y^2 - 2yz - z^2$ and $x_0 = 2e_1 + e_2 - 4e_3$, verify whether the function satisfies the hypothesis of implicit function theorem.

(b) Attempt any **two** : **4**

(1) Let ϕ be a function of class $C^{(2)}$ such that $\phi(x, f(x)) = 0$ and $\phi_2(x, f(x)) \neq 0$ for every x in R. Find f' and f".

(2) Let $\phi(f(y, z), y, z) = 0$ and $\phi_1(f(y, z), y, z) \neq 0$ for every (y, z) in R. Find f_{11}.

(3) Define manifold with a simple illustration.

(c) Attempt All : **3**

(1) Prove that (n – 1) sphere $\{x : |x| = 1\}$ is an (n – 1) manifold.

(2) Define the tangent vector to the manifold M at the point x_0.

(3) Define the normal vector to the manifold M at the point x_0.