Seat No. : _______________

JB2-101

January-2016

M.Sc., Sem.-I

405 : Mathematics
(Measure and Integration)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

   (1) Prove that a subset E of [a, b] is measurable if and only if for given $\in > 0$ there exist open sets G_1 and G_2 such that $G_1 \supseteq E$, $G_2 \supseteq E'$ and $|G_1 \cap G_2| < \in$.

   (2) Giving all the necessary details show that if G_1 and G_2 are open subsets of [a, b] such that $G_1 \subseteq G_2$, then $|G_1| \leq |G_2$.

   (b) Attempt any **two** : **4**

   (1) Let $E \subset [a, b]$. If $x \in E'$ and if $E \cup \{x\}$ is measurable then prove that E is measurable.

   (2) If $E \subseteq [a, b]$, then show that $\overline{m}E + \underline{m}E' = b - a$.

   (3) Using the definition of inner measure, show that the inner measure of the interval (1, 2) is 1.

   (c) Answer in brief : **3**

   (1) Give the characterization of the open subsets of [a, b] and hence define its length.

   (2) Give the definition of outer measure for a subset E of [a, b].

   (3) True or False : If F is a closed subset of [a, b] and $|F| = 0$, then the interior of F is an empty set.

2. (a) Attempt any **one** : **7**

   (1) If E_1 and E_2 are subsets of [a, b] then prove that

   $\overline{m}E_1 + \overline{m}E_2 \geq \overline{m}(E_1 \cup E_2) + \overline{m}(E_1 \cap E_2)$ and

   $\underline{m}E_1 + \underline{m}E_2 \leq \underline{m}(E_1 \cup E_2) + \underline{m}(E_1 \cap E_2)$.

   (2) Prove that non-measurable sets exist.

(b) Attempt any **two** : **4**

(1) Prove or disprove : Every measurable function on [a, b] is bounded.

(2) If E_1 and E_2 are measurable sets and $E_2 \subset E_1$, then show that $m(E_1 - E_2) = mE_1 - mE_2$.

(3) Prove that every constant function on [a, b] is measurable.

(c) Answer in brief : **3**

(1) If for the subsets A, B of [0, 2], mA = 1 and B = {1, 2}, then what is the measure of the set $A \cup B$?

(2) True or False : If $\overline{E}$ denotes the closure of E, then $m\overline{E} = mE$.

(3) State (only) a condition which is necessary but not sufficient for a function $f : [a, b] \to R$ to be measurable.

3. (a) Attempt any **one** : **7**

(1) If f and g are bounded measurable functions in L[a, b] then prove that f + g is in L[a, b] and moreover

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

(2) Let f be a bounded function in L[a, b]. If a < c < b, then prove that $f \in L[a, c] \cap L[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

(b) Attempt any **two** : **4**

(1) If f is bounded function in L[a, b] and if g is bounded function on [a, b] such that f = g a.e. then show that $g \in L[a, b]$ and further that

$$\int_a^b g = \int_a^b f.$$

(2) Let E_1 and E_2 denote the set of rationals and irrationals in [0, 1] respectively. If $f = \chi E_1 - \chi E_2$, then compute $\int_0^1 f$.

(3) Let $E_1 = [0, \pi/2] \cap Q$, $E_2 = [0, \pi/2] - E_1$ and $P = \{E_1, E_2\}$ be a measurable partition of $[0, \pi/2]$. If $f(x) = \sin x$ on $[0, \pi/2]$ then compute L(f, P).

(c) Answer in brief : **3**

(1) Let $f(x) = \sin^2 x$. If E and F are measurable subsets of [a, b] such that $E \subseteq F$, show that $\int_E f \leq \int_F f$.

(2) Give the definition of measurable partition of [a, b].

(3) Prove that the Dirichlet function is Lebesgue integrable on [0, 1].

4. (a) Attempt any **one** : **7**

(1) State and prove Lebesgue's dominated convergence theorem.

(2) State and prove Fatou's lemma and deduce the monotone convergence theorem.

(b) Attempt any **two** : **4**

(1) Using the absolute continuity of Lebesge integral show that if $f \in L[a, b]$ and if

$$g(x) = \int_a^x f(t)\, dt,$$

then g is uniformly continuous on [a, b].

(2) If a measurable function $f \in L[a, b]$ and $\lambda \in R$, then show that

$$\int_a^b \lambda f = \lambda \int_a^b f.$$

(3) If $f(x) = x - \sin x$, $0 \leq x \leq 2\pi$, then find f^+ and f^-.

(c) Answer in brief : **3**

(1) Explain what we mean by absolute continuity of the Lebesgue integral.

(2) Is product of two Lebesgue integrable functions on [a, b], a Lebesgue integrable function ?

(3) How do we find the Lebesgue integral of a non-negative measurable function on $(-\infty, \infty)$?

5. (a) Attempt any **one** : **7**

(1) Determine the Fourier series of the function $f(x) = |x|, -\pi \le x \le \pi$. Assuming that it converges everywhere, determine the value of $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

(2) Deriving all the necessary details, show that C[a, b] is dense in L_2[a, b].

(b) Attempt any **two** : **4**

(1) State and prove Minkowski's inequality.

(2) If $\{f_n\}$ is a sequence of functions in L_2[a, b] converging uniformly to $f \in L_2[a, b]$, then show that $\|f_n - f\|_2 \to 0$.

(3) Determine the Fourier series of the function $f(x) = 2\cos^2 x + \sin x \cos x$.

(c) Answer in brief : **3**

(1) True or False : $L_2[a, b] \subset C[a, b]$.

(2) How do we define a norm in L_2[a, b] ?

(3) What are the Fourier cosine coefficients of the function $f(x) = x^2 \sin x$?
