Seat No. : ______________

JA2-103

January-2016

M.Sc., Sem.-I

404 : Mathematics

(Ordinary Differential Equations)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Find the general solution of any **one** of the following equations in terms of power series in x : **7**

   (i) $y'' - y' + xy = 0$

   (ii) $y'' + (1 + x)\, y' - y = 0$

   (b) Attempt any **two** : **4**

   (i) Find the general solution of the equation $4y'' - 8y' + 7y = 0$

   (ii) Solve : $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 5$.

   (iii) Solve $y' = -2xy$ in terms of a power series in x.

   (c) Answer very briefly : **3**

   (i) Solve : $xy' - 3y - x^4 = 0$

   (ii) What is the radius of convergence of the series $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n x^{2n}$?

   (iii) Verify that $y_1 = x$ is one solution of $x^2y'' + 2xy' - 2y = 0$ and find y_2.

2. (a) Verify that the origin is a regular singular point and calculate two independent Frobenius series solutions for the equation $2x^2y'' + xy' - (x + 1)y = 0$. **7**

   OR

   Show that the equation $4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0$ has only one Frobenius series solution. Find the general solution.

(b) Attempt any **two** : **4**

(i) Find the indicial equation and its roots for the equation.

$x^3y'' + (\cos 2x - 1)y' + 2xy = 0.$

(ii) Find the general solution near $x = 0$ of the hypergeometric equation

$x(1 - x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0.$

(iii) What is the nature of the point $x = \infty$ for the equation

$xy'' + (1 - x)y' - y = 0$? Justify your answer.

(c) Answer very briefly : **3**

(i) Is $x = 1$ a regular singular point of the equation $(x - 1)^2y'' + xy' + y = 0$?

(ii) Write down the general solution of the Gauss's hypergeometric equation near the singular point $x = 1$.

(iii) Determine the nature of the point $x = 0$ for the equation $xy'' + (\cos x)\, y = 0$.

3. (a) Derive the recursion formula for the Chebyshev polynomials and obtain $T_2(x)$, $T_3(x)$ and $T_4(x)$ by taking $T_0(x) = 1$and $T_1(x) = x$. **7**

OR

Prove that $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ if $m \neq n$.

(b) Attempt any **two** : **4**

(i) Show that $T_n(x) = \frac{1}{2}\left[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\right]$

(ii) Find the first two terms of the Legendre series of $f(x) = e^x$.

(iii) Calcualte Legendre polynomial $P_2(x)$.

(c) Answer very briefly : **3**

(i) What is the value of $\int_{-1}^{1} \frac{[T_4(x)]^2}{\sqrt{1-x^2}}\, dx$?

(ii) Write down the recursion formula for the Legendre polynomials.

(iii) State the Minimax property of Chebyshev polynomials. (Do not prove).

4. (a) Find the general solution of the Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$, where p is a non-negative constant and is not an integer. **7**

OR

Express $J_2(x)$, $J_3(x)$, and $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

(b) Attempt any **two** : **4**

(i) Show that $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$.

(ii) Show that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

(iii) Show that $1 = J_0(x) + 2J_2(x) + 2J_4(x) + \ldots$..

(c) Answer very briefly : **3**

(i) Show that between any two positive zeros of $J_0(x)$ there is a zero of $J_1(x)$.

(ii) What is the value of $\Gamma(4)$?

(iii) What is the value of $J_{1/2}(\pi)$?

5. (a) Consider the initial value problem **7**

$y' = x^2 - y$, $y(0)= 0$.

Find successive approximations $y_0(x)$, $y_1(x)$, $y_2(x)$, $y_3(x)$ using Picard's method.

OR

Find the third approximation of the solution of the following initial value problem by Picard's method :

$$\begin{cases} \dfrac{dy}{dx} = z, & y(0) = 1 \\ \dfrac{dz}{dx} = x^3(y+z), & z(0) = \dfrac{1}{2} \end{cases}$$

(b) Attempt any **two** : **4**

(i) State (only) Picard's theorem.

(ii) Show that $f(x,y) = xy^2$ satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$.

(iii) Does $f(x,y) = y^{1/2}$ satisfy a Lipschitz condition on the rectangle $|x| \le 1$ and $0 \le y \le 1$? Justify your answer.

(c) Answer very briefly : **3**

(i) State Lipschitz condition.

(ii) Does $f(x,y) = xy$ satisfy a Lipschitz condition on any rectangle $a \leq x \leq b$ and $c \leq y \leq d$.

(iii) Let $f(x, y)$ be a continuous function on the closed rectangle R. If f does not satisfy the Lipschitz condition on R, then what can you say about the solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$?