Seat No. : _______________

NF-131

December-2015

M.Sc., Sem.-III

503 : Mathematics

(Number Theory)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one**. **7**
   - (i) Prove that Fermat numbers are all relatively prime to each other. Using this result prove that the number of primes is infinite.
   - (ii) Find integers x, y, z such that $35x + 55y + 77z = 1$.

   (b) Attempt any **two**. **4**
   - (i) Prove that if a and b are integers, with $b > 0$, then there exist unique integers q and r satisfying $a = qb + r$, where $2b \leq r < 3b$.
   - (ii) Prove that the sum of the squares of two odd integers cannot be a perfect square.
   - (iii) Find all pairs of primes p and q satisfying $p - q = 3$.

   (c) Answer very briefly. **3**
   - (i) Prove or disprove : If $a|(b + c)$, then either $a|b$ or $a|c$.
   - (ii) Is the integer 701 prime ? Justify your answer.
   - (iii) State the fundamental theorem of arithmetic.

2. (a) Attempt any **one**. **7**
   - (i) Prove that every even perfect number is of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is a prime.
   - (ii) State and prove the Möbius inversion formula.

   (b) Attempt any **two**. **4**
   - (i) Let x and y be real numbers. Prove that $[x] + [y] \leq [x + y]$.
   - (ii) For $n = 3655$, find the values of $\tau(n)$, $\tau(n + 1)$ and $\tau(n + 2)$.
   - (iii) Prove that the product of two odd primes is never a perfect number.

   (c) Answer very briefly. **3**
   - (i) For what real numbers x is it true that $[x + 3] = 3 + [x]$?
   - (ii) Find the highest power of 5 dividing 15000!.
   - (iii) Calculate ϕ (5225).

3. (a) Attempt any **one**. **7**
   - (i) Using the theory of indices, solve $3x^4 \equiv 5 \pmod{11}$.
   - (ii) State and prove the Lagrange's theorem.

   (b) Attempt any **two**. **4**
   - (i) Solve : $2x \equiv 1 \pmod 5$, $3x \equiv 9 \pmod 6$, $4x \equiv 1 \pmod 7$, $5x \equiv 9 \pmod{11}$.
   - (ii) Solve : $140x \equiv 133 \pmod{301}$.
   - (iii) For $k \geq 3$ prove that the integer 2^k has no primitive roots.

(c) Answer very briefly. **3**

(i) Find the unit digit of 3^{100}.

(ii) Is the converse of Wilson's theorem true ? Justify your answer.

(iii) What is the order of the integer 9 modulo 13 ?

4. (a) Attempt any **one**. **7**

(i) Determine the general solution of $364x + 227y = 1$ by means of simple continued fractions.

(ii) Prove that the value of any infinite continued fraction is an irrational number.

(b) Attempt any **two**. **4**

(i) Determine the infinite continued fraction representation of $\sqrt{23}$.

(ii) Evaluate $[2;\overline{1,2,1}]$

(iii) Obtain all primitive Pythagorean triples of the form 40, y, z.

(c) Answer very briefly. **3**

(i) Express $\frac{118}{303}$ as finite simple continued fraction.

(ii) Compute the convergents of [1; 2, 3, 3, 2, 1].

(iii) What is the fundamental solution of $x^2 - 11y^2 = 1$?

5. (a) Attempt any **one**. **7**

(i) Determine all algebraic integers of the field $\mathbb{Q}(\sqrt{m})$ where m is a square-free rational integer, positive or negative but not equal to 1.

(ii) Show that the fields $\mathbb{Q}(\sqrt{m})$ for m = –1, –2, –3, –7, 2, 3 are Euclidean.

(b) Attempt any **two**. **4**

(i) Prove that the norm of an integer in $\mathbb{Q}(\sqrt{m})$ is a rational integer.

(ii) Prove that if the norm of an integer α in $\mathbb{Q}(\sqrt{m})$ is $\pm$ p, where p is a rational prime, then α is a prime.

(iii) Let $\mathbb{Q}(\sqrt{m})$ have the unique factorization property. Then prove that to any prime π in $\mathbb{Q}(\sqrt{m})$ there corresponds one and only one rational prime p such that $\pi | p$.

(c) Answer very briefly. **3**

(i) Prove that reciprocal of a unit is a unit.

(ii) Is $11 + 2\sqrt{6}$ a prime in $\mathbb{Q}(\sqrt{6})$?

(iii) If α is any integer, and u any unit, in $\mathbb{Q}(\sqrt{m})$, prove that $u | \alpha$.