Seat No. : _______________

# NB-134

**December-2015**

**M.Sc., Sem.-III**

**501 : Mathematics**

**(Functional Analysis – I)**

**Time : 3 Hours]** **[Max. Marks : 70**

1. (A) Attempt any **one** : **7**

   (1) Let M and N be subspaces of a vector space V, such that V = M + N. Show that $V = M \oplus N$ if and only if $M \cap N = \{0\}$.

   (2) Show that a linear map $T : V \to V$ is non-singular if and only if T (B) is a basis whenever B is a basis.

   (B) Attempt any **two :** **4**

   (1) Let $f(x) = x^2$ and $g(x) = \cos x$. Prove or disprove that S = {f, g} is a linearly dependent subset of C[0, 1].

   (2) Find a linear transformation $T : R^2 \to R^2$ such that T(1, 1) = (2, 0) and T(0, 1) = T(1,0).

   (3) Let $V = M \oplus N$ and E be a projection on M along N. Show that E is idempotent.

   (C) Answer in brief : **3**

   (1) Prove or disprove: C[0, 1] is finite dimensional.

   (2) Let V denote the space of all real-valued polynomials with real coefficients and S denote the set of all polynomials of degree 3. Then what is the span of S in V ?

   (3) If A is an algebra with identity then show that every ring ideal in A is also an algebra ideal in A.

2. (A) Attempt any **one** : **7**

   (1) If M is a closed subspace of a Banach space N, then show that N/M is a Banach space with respect to the norm defined by

   $\| x + M \| = \inf \{\| x + m \| : m \in M\}$.

   (2) Prove that every linear transformation on $l_\infty^n$ is continuous. Is this true in case of $l_\infty$ ? Justify your answer.

(B) Attempt any **two :** **4**

(1) Prove that norm is a continuous function.

(2) For a bounded linear map T, show that

$$\sup\{\| T(x) \| : \| x \| \leq 1\} = \sup\{\| T(x) \| : \| x \| = 1\}$$

(3) Let T : N → N' be a continuous linear map and let M be its Kernel. Then show that the mapping F : N/M → N' defined by F(x + M) = T(x) is bounded and $\| F \| = \| T \|$.

(C) Answer in brief. **3**

(1) If $(x_n)$ and $(y_n)$ are convergent sequences in a normed linear space N, then show that $(x_n + y_n)$ is also a convergent sequence in N.

(2) Define a bounded linear map.

(3) Show that there exist positive numbers $k_1$ and $k_2$ such that

$$k_1 \| x \|_\infty \leq \| x \|_1 \leq k_2 \| x \|_\infty, \text{ for all } x \in R^n.$$

3 (A) Attempt any **one :** **7**

(1) If N* is separable, then prove that N is separable.

(2) State Hahn-Banach theorem and use it to show that if $x$ is a non-zero vector of N then there exits $f \in N^*$ such that $f(x) = \| x \|$ and $\| f \| = 1$. Further, deduce that $x \to F_x$ is a norm preserving map of N into N**.

(B) Attempt any **two :** **4**

(1) State the conjugate spaces of $l_1$, $c_0$, $l_2^n$ and $l_\infty^n$.

(2) If M is a closed subspace of N and $x \notin M$, then show that there exists $f \in N^*$ such that $f(M) = 0$ and $f(x) \neq 0$.

(3) Show that $l_1$ is separable.

(C) Answer in brief : **3**

(1) Show that the mapping $x \to F_x$ is linear.

(2) What is the dimension of $\left(l_2^3\right)^*$?

(3) True or False : Every reflexive space is complete.

4. (A) Attempt any **one :** **7**

(1) State and prove the closed graph theorem.

(2) State the Uniform Boundedness theorem. Use it to prove that a subset X of N is bounded if and only if f(X) is a bounded set for each $f \in N^*$.

(B) Attempt any **two :** **4**

(1) Let $T : B \to B$ be linear, bounded, one-one and onto. Assuming the closed graph theorem prove that T is open.

(2) If a Banach space B is reflexive, then show that B* is also reflexive.

(3) If $T \in B(N)$, then show that the mapping $T \to T^*$ is linear.

(C) Answer in brief : **3**

(1) State the open mapping theorem.

(2) For $T \in B(N)$, define its conjugate T* and show that it is linear.

(3) Show that the conjugate of an identity operator is an identity operator.

5. (A) Attempt any **one :** 7

(1) Prove that a non-empty closed and convex subset C of H has a unique vector of the smallest norm. Show that the result fails if C is not closed or not convex.

(2) Show that a Hilbert space H is finite dimensional if and only if every complete orthonormal set in H is a Hamel basis.

(B) Attempt any **two :** **4**

(1) Let $S = \{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in H. Then prove that the following statements are equivalent :

(a) S is complete;

(b) $x \perp S \Rightarrow x = 0$;

(c) if $x \in H$, then $x = \Sigma_{i=1}^{n} \langle x, e_i \rangle e_i$;

(d) if $x \in H$, then $\| x \|^2 = \Sigma_{i=1}^{n} | \langle x, e_i \rangle |^2$

(2) If $S = \{e_1, e_2, \ldots, e_n\}$ is an orthonormal set in H, then show that for every $x \in H$, $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp S$.

(3) State and prove the parallelogram law in a Hilbert space.

(C) Answer in brief : **3**

(1) State Bessel's inequality.

(2) Let S be a non-empty subset of H. Show that $S^{\perp}$ is a closed subspace of H.

(3) Let S be a non-empty subset of H. Show that $S \subseteq S^{\perp\perp}$.