Seat No. : _______________

ZA-104

April-2014

M.Sc. Sem.-IV

Mathematics

MAT-507 : (Functional Analysis – II)

Time : 3 Hours] **[Max. Marks : 70**

Instruction : Each question carries equal **14** marks.

1. (a) Attempt any **one** : **7**

(1) Prove that the set of all self-adjoint operators in B(H) form a closed real linear subspace of B(H).

(2) If A is a positive operator, prove that I + A is invertible.

(b) Attempt any **two** : **4**

(1) Prove that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents a normal operator on $\mathbb{R}^2$ if and only if b = c or else b = – c, a = d.

(2) Prove that $O \leq A \leq B$ does not imply that $A^2 \leq B^2$.

(3) Prove that $T \in B(H)$ is self-adjoint if and only if $<Tx, x>$ is real for all $x \in H$.

(c) Attempt **all** : **3**

(1) If $T \in B(H)$ is self-adjoint and non-zero, prove that T^2 is also self-adjoint and non-zero.

(2) Prove or disprove : Product of two positive operators is positive.

(3) If H is a Hilbert space, prove H* is also a Hilbert space.

2. (a) Attempt any **one** : **7**

(1) If P is a projection on H with range M and nullspace N, then prove that $M \perp N$ if and only if P is self-adjoint, in this case, $N = M^{\perp}$.

(2) If P is a projection with range M, prove that $x \in M \Leftrightarrow Px = x \Leftrightarrow \| Px \| = \| x \|$.

(b) Attempt any **two** : **4**

(1) If A and B are self-adjoint, then prove that AB = 0 if and only if range of A is orthogonal to the range of B.

(2) Prove that the set of all unitary operators in B(H) is closed in B(H).

(3) If T is unitary, prove that T is onto and $R(\overline{T}) = H$.

(c) Attempt **all** : **3**

(1) Give an example of a normal operator that is not unitary.

(2) Give an example of a self-adjoint operator that is not positive.

(3) Give an example of an isometry that is not invertible.

3. (a) Attempt any **one** : **7**

(1) Let $T \in B(H)$ be normal with spectrum $\{\lambda_1, \lambda_2, \ldots . \lambda_m\}$.

Prove that T is positive $\Leftrightarrow \lambda_i \geq 0$ for each i and

T is unitary $\Leftrightarrow |\lambda_i| = 1$ for each i.

(2) If $T^K = 0$ for some positive integer k then prove that $\sigma(T) = \{0\}$.

(b) Attempt any **two** : **4**

(1) For a non-singular $T \in B(H)$, prove that $\lambda \in \sigma(T) \Leftrightarrow \lambda^{-1} \in \sigma(T^{-1})$.

(2) Define similar matrices, prove that matrices

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ are similar.

(3) Find an operator on $\mathbb{R}^2$ whose eigen spectrum σ_e is empty.

(c) Attempt **all** : **3**

(1) Find a linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\sigma(T) = \left\{\frac{1}{2}\right\}$.

(2) Describe the spectrum of $T^K + 1$ in terms of the spectrum of T.

(3) Give an $n \times n$ matrix A such that $A^{n-1} \neq 0$, but $A^n = 0$.

4. (a) Attempt any **one** : **7**

(1) If X is a normed linear space and $A \in B(X)$ is of finite rank, then prove that $\sigma_e(A) = \sigma(A)$.

(2) If X is a Banach space then prove that the set of all regular elements in B(X) is open in B(X) and the mapping $A \rightarrow A^{-1}$ is continuous.

(b) Attempt any **two** : **4**

(1) Find the spectrum of $A : l^p \rightarrow l^p$ $(1 \leq p \leq \infty)$ given by $A(x_1, x_2, x_3, \ldots..) = (x_1, x_{2/2}, x_{3/3}, \ldots..)$.

(2) Find an operator T on C[0, 1] such that $\sigma(T) = [0, 1]$.

(3) If A is the right shift operator on l^2, find the approximate eigen spectrum $\sigma_a(A)$.

(c) Attempt **all** : **3**

(1) Give statement (only) of Gelfand Mazur theorem, spectral radius formula.

(2) Does there exist $T \in B(H)$ such that $\sigma(T) = \mathbb{R}$? Justify your answer.

(3) Give an example of a projection on a Hilbert space whose spectrum is $\{0, 1\}$.

5. (a) Attempt any **one** : **7**

(1) If F and G are compact linear operators on X, prove that F + G is also compact linear.

(2) Let X and Y be normal linear spaces and $F : X \to Y$ be linear.

Prove that F is compact $\Leftrightarrow$ for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

(b) Attempt any **two** : **4**

(1) Define compact, linear map on a normed linear spaces. Give two examples of compact linear maps.

(2) If X is infinite dimensional normed linear space and A is a compact operator in X, prove that $O \in \sigma_a(A)$.

(3) Define the term : bounded below. If $T \in B(X)$ is bounded below, prove that T is injective.

(c) Attempt **all** : **3**

(1) Give an example of a bounded, linear operator that is not compact. Justify your answer.

(2) Prove or disprove $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (y, x)$ is compact, linear map.

(3) Prove that every compact, linear map is a bounded, linear map.
