Seat No. : _______________

XX-136

M.Sc. (Sem. IV) (CBCS)
April-2013

507 – Mathematics

(Functional Analysis-II)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (1) Each question carries equal **14** marks.
(2) Follow usual notations.

1. (a) Let H be a Hilbert space and $T \in BL(H)$, prove that there is a unique mapping T* from H into H (called the adjoint of T) which satisfies the relation $<Tx, y> = <x, T^*y>$ for all $x, y \in H$. **7**

OR

Prove that the adjoint operation $T \to T^*$ on BL(H) has the following properties :

(i) $\| T^* \| = \| T \|$

(ii) $\| T^*T \| = \| T \|^2$

(b) Attempt any **two :** **4**

(1) If $T \in BL(H)$ is self-adjoint and $T \neq 0$, then prove that $T^n \neq 0$ for n = 2, 4, 8, 16, ….

(2) Show that any $T \in BL(H)$ can be uniquely written as $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint.

(3) If T is a self-adjoint operator on H, show that $\| T \| = \sup \left\{ \frac{| < T_x, x> |}{\| x \|^2} / x \neq 0 \right\}$.

(c) Attempt **all** : **3**

(1) Let H be a Hilbert space over $\mathbb{R}$ and $T \in BL(H)$. Show that T is self-adjoint $\Leftrightarrow <Tx, y> = <Ty, x>$ for all $x \in H$.

(2) Let $H = \mathbb{C}^3$ and $A \in BL(H)$ defined by the matrix $M = \begin{pmatrix} 1 & a & 0 \\ i & 1 & 0 \\ 0 & 0 & i \end{pmatrix}$. Find values of a, if any, such that A is self-adjoint.

(3) If A, B are self-adjoint, prove that AB is self-adjoint $\Leftrightarrow AB = BA$.

2. (a) Prove that the set of all normal operators in BL(H) is a closed subset of BL(H). Is it a linear subspace of BL(H) ? **7**

OR

Define positive operator. If A is a positive operator on H, prove that I + A is invertible.

(b) Attempt any **two :** **4**

(1) Show that an isometric operator $T \in BL(H)$ which is not unitary maps H onto a proper closed subspace of H.

(2) If P and Q are projections on closed linear subspaces M and N respectively. Then prove that following are equivalent :

(i) PQ = P

(ii) $M \subseteq N$

(iii) $P \leq Q$

(3) If M is a closed linear subspace of H and M is invariant under $T \in BL(H)$, then prove that $M^{\perp}$ is invariant under T^*.

(c) Attempt **all :** **3**

(1) Give an operator on $\mathbb{R}^2$ that is self-adjoint but not positive.

(2) If P and Q are projections, under which condition P + Q is also a projection ?

(3) If $U \in BL(H)$ is unitary, what is the value of $\| U \|$?

3. (a) Show that the mapping $T \to [T]$ is a one one onto map from the set of all operators onto the set of all matrices. **7**

OR

If H is a finite dimensional complex Hibert space and $T \in BL(H)$, prove that $\sigma(T)$ is non-empty.

(b) Attempt any **two :** **4**

(1) If $T \in BL(H)$,then prove that $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$.

(2) If $T^k = 0$ for some k then show that $\sigma(t) = \{0\}$.

(3) Give an operator on $\mathbb{R}^2$ whose spectrum is empty.

(c) Attempt **all** : **3**

(1) Give an operator on $\mathbb{C}^2$ whose spectrum is {i}.

(2) Give three square roots of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(3) If T is normal then prove that x is an eigen vector of T with respect to eigen value λ then x is an eigen vector of T* with respect to eigen value $\bar{\lambda}$.

4. (a) Let X be a Banach space and $A \in BL(X)$. Prove that A is invertible if A is bounded below and the range of A is dense in X. **7**

OR

Let X be a Banach space over K and $A \in BL(X)$. Prove that the spectrum $\sigma(A)$ is a compact subset of K.

(b) Attempt any **two :** **4**

(1) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (0, y)$. Find $\sigma(T)$.

(2) Let X = C[a, b] with sup norm. For a fix $x_0 \in X$ and $x \in X$, define $A(x) = x_0 x$. Find the eigen spectrum $\sigma_e(A)$ of A.

(3) Find the spectrum $\sigma(T)$ of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x, 2y)$

(c) Attempt **all** : **3**

(1) Define approximate eigen spectrum $\sigma_a(A)$ of $A \in BL(X)$. Give a characterization (without proof) of approximate eigen values of A.

(2) If A is of finite rank, then prove that $A - KI$ is invertible $\Leftrightarrow$ $A - KI$ is one-one.

(3) Find the spectrum of the zero operator and the identity operator on X. (Here, X is a non-zero normed linear space).

5. (a) Prove that A $\in$ BL(X) is a compact linear map if and only if for every bounded sequence (x_n) in X, $(T(x_n))$ has a convergent subsequence. **7**

OR

If X is a normed linear space and A $\in$ CL(X) (that is A is compact map) then prove that $0 \in \sigma_a(A)$ whenever X is infinite dimensional.

(b) Attempt any **two :** **4**

(1) If A and B are compact linear maps on a normed linear space X, prove that A + B is compact.

(2) Prove that any functional on a normed linear space is compact.

(3) Give an operator A $\in$ BL(X) such that $\sigma(A) \neq \sigma(A')$, where A' denote the transpose of A.

(c) Attempt **all :** **3**

(1) If X is an infinite dimensional normed linear space, for what values of α, αI is compact ?

(2) Is A : $l^2 \to l^2$ defined by $A(e_n) = e_{n+1}$ compact ? Why ?

(3) True or false : CL(X) is closed in BL(X). Give reason.
