Seat No. : _______________

XA-130

T.Y.B.Sc.

March-2013

Mathematics : Paper – VII

(Real Analysis – I)

(New Course)

Time : 3 Hours] **[Max. Marks : 105**

Instructions : (1) **All** questions are compulsory.

(2) In each question, C Part is of short and is compulsory.

(3) Symbols used have their usual meaning.

(4) Each question is of **21** marks.

1. (a) Define characteristic function of $A \subset X$. If $A, B \subset X$ then prove that

(i) $X_A(x) \leq X_B(x) \Leftrightarrow A \subset B$.

(ii) $X_{A \cup B}(x) = X_A(x) + X_B(x) - X_{A \cap B}(x)$

OR

If A and B are the two non-empty subsets of R which has a least upper bound. If $C = \{a + b / A \in A, b \in B\}$ then prove that C has the least upper bound and lub C = lubA + lubB.

(b) Prove that [0, 1] is uncountable.

OR

If A is the set of all sequences whose elements are the digits 0 and 1 then prove that A is uncountable.

(c) (i) Let X be any subset of R. If μ = lubX then prove that every $\varepsilon > 0 \ \exists \ x \in X \in \mu - \varepsilon < x \leq \mu$.

(ii) If x, y, $\in$ R, $x < y$ then prove that $\exists r \in \phi \ \exists \ x < r < y$.

2. (a) Prove that $\lim_{x \to a} f(x) = L \Leftrightarrow f(x_n) \to L$ for every sequence (x_n) in the domain of f with $x_n \to a$ (n = 1, 2, 3, …)

OR

Let f and g be continuous real valued functions. If f is continuous at a and g is continuous at f(a) then prove that gof is continuous at a.

(b) State and prove a fixed point theorem

OR

If f is monotonic on (a, b) then prove that for each $C \in (a, b)$, $\lim_{x \to c_+} f(x)$ and $\lim_{x \to c_-} f(x)$ both exist.

(c) (i) Discuss different kinds of discontinuity of a real valued function f on R.

(ii) If f is continuous at $x = a$ then prove that | f | is continuous at $x = a$.

3. (a) Define an open sphere in a metric space (x, d). Prove that every open sphere in (x, d) is an open set.

OR

Define : A metric space. Prove that (R^2, d) is a metric space where $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2) \in R^2$, $y = (y_1, y_2) \in R^2$.

(b) In any metric space (x, d). Prove that the intersection of a finite number of open sets is open. Is it true for an infinite intersection ? Justify your answer.

OR

Define : A closed set in a metric space (x, d). Prove that a subset F of a metric space x is closed $\Leftrightarrow$ its complement F' is open.

(c) (i) Define : Homomorphism. If $f : X_1 \to X_2$ is homomorphism then prove that f^{-1} is also homomorphism on X_2.

(ii) Which of the following set in R are open or closed or not open and not closed ?

(a) Q

(b) {1, 2, 3}

(c) (0, 1]

(d) [a, b]

4. (a) Define compact metric space. Prove that the metric space (x, d) is compact $\Leftrightarrow$ every sequence of points in x has a subsequence converging to a point in x.

OR

Let (x, d) be a complete metric space. If T is a contraction in x then prove that $T_x = x$ has a unique solution for x.

(b) If the subset A of the metric space (x, d) is totally bounded then prove that A is bounded.

OR

Prove that a metric space (x, d) is disconnected $\Leftrightarrow$ X is the union of two non-empty disjoint open sets.

(c) (i) Prove that every finite subset E of any metric space (X, d) is compact.

(ii) If t : X $\rightarrow$ X is defined by $T_x = x$, where $X = \left[0, \frac{1}{3}\right]$ then prove that T is a contraction on $\left[0, \frac{1}{3}\right]$.

5. (a) State and prove the inverse function theorem.

OR

State and prove Cauchy's mean value theorem and hence deduce Lagrange's Mean value theorem.

(b) State and prove Taylor's theorem.

OR

Define : Derivative of function at a point. If f is a differentiable function at a point a $\in$ I, ICR then prove that f is continuous at a.

(c) (i) Prove that between any two real roots of equation $e^x.\sin x = 1$, there is at least one real root of $e^x.\cos x + 1 = 0$.

(ii) Evaluate : $\lim_{x \rightarrow 1} \dfrac{x^x - x}{x - 1 - \log x}$
