3010E946 Candidate's Seat No :________

M.Sc. Sem.-4 (Rep) Examination

508 - Mathematics

October-2025

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Let R be a commutative ring with unity and let A be an ideal of R. Show that R/A is an integral domain if and only if A is prime. 7

   (B) If n is an integer greater than 1, show that $\langle n \rangle = n\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ if and only if n is prime. 7

OR

1. (A) State and prove first isomorphism theorem for rings. 7

   (B) Let R be a commutative ring with unity. Suppose that the only ideals of ring R are $\{0\}$ and R. Show that R is a field. Does the converse hold? Explain. 7

2. (A) Let F be a field and let $p(x) \in F[x]$. Prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F. 7

   (B) State (without proof) mod p irreducibility test. Discuss the irreducibility of the polynomial $f(x) = x^5 + 2x + 4$ over $\mathbb{Q}$. 7

OR

2. (A) In a principal ideal domain, prove that an element is an irreducible if and only if it is prime. 7

   (B) Define Euclidean domain.
   Show that the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain. 7

3. (A) Define splitting field of a polynomial $f(x)$ over a field F.
   Find the splitting field E of $x^4 + x^2 + 1$ over $\mathbb{Q}$. Find the degree $[E : \mathbb{Q}]$. 7

   (B) Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Prove that there is an extension field E of F in which $f(x)$ has a zero. 7

OR

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3. (A) If K is an algebraic extension of E and E is an algebraic extension of F, prove that K is an algebraic extension of F. 7

(B) Draw the subfield lattices of $\mathrm{GF}(3^{18})$ and $\mathrm{GF}(2^{30})$. 7

4. (A) Let $E = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Find the Galois group $\mathrm{Gal}(E/\mathbb{Q})$. Discuss the lattice of subgroups of $\mathrm{Gal}(E/\mathbb{Q})$ and the lattice of subfields of E. 7

(B) Is the polynomial $g(x) = 3x^5 - 15x + 5$ solvable by radicals over $\mathbb{Q}$? Explain. 7

OR

4. (A) Define cyclotomic polynomial $\Phi_n(x)$. Find $\Phi_n(x)$, for $n = 1, 2, 3, 4, 5, 6$. 7

(B) Determine the Galois group of $x^3 - 2$ over $\mathbb{Q}$. 7

5. **Attempt any seven of the following.** 14

(1) Consider the ring $\mathbb{Z}_{10}$ and its subring $S = \{0, 2, 4, 6, 8\}$. Which of the following statements are true?

(A) $\mathbb{Z}_{10}$ and S have different unity.

(B) $\mathbb{Z}_{10}$ and S have the same unity.

(C) $\mathbb{Z}_{10}$ has zero-divisors but S has no zero-divisors.

(D) S is a field under addition and multiplication modulo 10.

(2) The number of zero-divisors of the ring $\mathbb{Z}_{14}$ is

(A) 2 (B) 6 (C) 7 (D) 8

(3) Which of the following integers are units in the ring $\mathbb{Z}_{15}$?

(A) 1 (B) 4 (C) 8 (D) 9

(4) The number of zeros of the polynomial $x^2 + 3x + 2$ in the ring $\mathbb{Z}_6$ is

(A) 2 (B) 3 (C) 4 (D) 5

(5) Consider the polynomials $f(x) = x^2 + x + 4$ and $g(x) = x^2 + 2x + 3$. Then

(A) $f(x)$ and $g(x)$ both are reducible over $\mathbb{Z}_5$.

(B) $f(x)$ and $g(x)$ both are irreducible over $\mathbb{Z}_5$.

(C) $f(x)$ is irreducible over $\mathbb{Z}_5$ but $g(x)$ is reducible over $\mathbb{Z}_5$.

(D) $f(x)$ is reducible over $\mathbb{Z}_5$ but $g(x)$ is irreducible over $\mathbb{Z}_5$.

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(6) In $\mathbb{Z}[\sqrt{-5}]$, $1+3\sqrt{-5}$ is

(A) irreducible and prime.
(B) irreducible but not prime
(C) prime but not irreducible
(D) neither irreducible nor prime.

(7) The splitting field of the polynomial $x^4 - x^2 - 2$ over $\mathbb{Q}$ is

(A) $\mathbb{Q}(\sqrt{2}, i)$
(B) $\mathbb{Q}(i)$
(C) $\mathbb{Q}(\sqrt{2})$
(D) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

(8) The minimal polynomial for $\sqrt{2} - \sqrt{3}$ over $\mathbb{Q}$ is

(A) $x^4 + 10x^2 + 1$
(B) $x^4 - 10x^2 - 1$
(C) $x^4 - 10x^2 + 1$
(D) $x^4 - x^2 + 1$

(9) $[GF(1024) : GF(32)] =$ ________

(A) 5 (B) 2 (C) 10 (D) 32

(10) Which of the following is **not** a field?

(A) $Z_5[x]/\langle x^2+1\rangle$
(B) $Z_2[x]/\langle x^2+x+1\rangle$
(C) $Z_3[x]/\langle x^2+1\rangle$
(D) $Q[x]/\langle x^2+1\rangle$

(11) The order of the Galois group of the field $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ is

(A) 6 (B) 3 (C) 1 (D) 2

(12) Let $\alpha = \cos(2\pi/15) + i\sin(2\pi/15)$ Then $\text{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is isomorphic to

(A) $\mathbb{Z}_8$
(B) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$
(C) $\mathbb{Z}_{15}$
(D) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
