Seat No. : _______________

MN-205

May-2025

B.Sc., Sem.-IV

MAT-205 : Mathematics

(Abstract Algebra – I)

(Old Course)

Time : 2:30 Hours] **[Max. Marks : 70**

Instructions : (i) **All** questions are compulsory.

(ii) Write the question number in your answer sheet as shown in the question paper.

(iii) Figures to the right indicate marks of the question.

1. (A) Define a congruence relation. Prove that "congruence modulo n" is an equivalence relation. **7**

1. (B) Prove that a group G is commutative if $(ab)^i = a^i b^i$ for all a, b in G, for any three consecutive integers. **7**

OR

1. (A) Show that $G = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \text{ and } a^2 + b^2 \neq 0 \right\}$ forms a commutative group under matrix multiplication. **7**

1. (B) Define group and prove that every group has unique identity element, and unique inverse for any element of group. **7**

2. (A) State and prove Lagrange's theorem for subgroup H of finite group G. **7**

2. (B) Prove that if H is subgroup of G, then the set $x^{-1}Hx = \{x^{-1}hx \mid h \in H\}$ is a subgroup of G for $x \in G$. **7**

OR

2. (A) Prove that finite non-empty subset H of group G is a subgroup of G if it is closed under multiplication. **7**

2. (B) If G and H are group under same binary operation then show that $G \cap H$ is always group under same binary operation. What about $G \cup H$? Is it group under same binary operation ? Justify your answer. **7**

3. (A) Check whether the following elements of S_8 are even permutation or odd permutation. **7**

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

3. (B) List all permutations on S = {1, 2, 3} and form a group table under binary operation composition, is this group commutative ? **7**

OR

3. (A) Prove that a subgroup H of G is normal subgroup of G if and only if $aHa^{-1} \subset H$ for each $a \in G$. **7**

3. (B) Define quotient group. If G is a cyclic group of order 12 generated by a, then $G = \{e, a, a^2,.., a^{11} \mid a^{12} = e\}$ take $H = \{e, a^3, a^6, a^9\}$ which is normal subgroup of G. List all the elements of quotient group G/H, also form a group table of G/H. **7**

4. (A) State and prove first fundamental theorem of homomorphism. **7**

4. (B) Prove that an isomorphism between two groups is an equivalence relation. **7**

OR

4. (A) Define kernel of group homomorphism. Also prove that the kernel K_ϕ of a homomorphism $\phi : (G ; o) \to (G ; *)$ is a normal subgroup of G. **7**

4. (B) Prove that, A cyclic group G with generator a is finite if and only if there exists a positive integer k such that $a^k = e$. **7**

5. Give the answer in brief : (Any **seven**) **14**

(1) Define transitive relation, also give an example of transitive relation.

(2) Define commutative group and also give an example of commutative group.

(3) Define reflexive relation with one example.

(4) Define left-coset and right-coset.

(5) Draw a lattice diagram of subgroups of $(Z_8, +_8)$.

(6) State Euler's theorem.

(7) Express permutation (1, 3, 4, 6, 7, 9) as product of transpositions, hence deduce that is it odd permutation or even permutation.

(8) Define normal subgroup also give one example of normal subgroup of any group.

(9) Define even and odd permutation.

(10) Define isomorphism and automorphism of group.

(11) Define kernel of homomorphism, also give one example of it.

(12) State Cayley's theorem.

Seat No. : _______________

MN-205

May-2025

B.Sc., Sem.-IV

MAT-205 : Mathematics

(Ring Theory)

(New Course)

Time : 2:30 Hours] **[Max. Marks : 70**

Instructions : (i) **All** questions are compulsory.
(ii) Write the question number in your answer book as shown in the question paper.
(iii) The figures to the right indicates marks of the question.

1. (A) Show that the set $A = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in \mathbb{Z} \right\}$ forms an integral domain under usual addition and multiplication of matrices. **7**

1. (B) Define a ring with unity. If a R is a ring with unity 1 then prove the following properties in R. (* is multiplication) **7**
   (1) $a * (-b) = (-a) * b = -(a * b), \ \forall a, b \in R$
   (2) $(-1) * (-1) = 1$

OR

1. (A) Define a Boolean ring and prove that a Boolean ring is a commutative ring. Also give an example of Boolean ring. **7**

1. (B) Prove that set $F = \{a + b\sqrt{2} \ ; \ a, b \in \mathbb{Q}\}$ is a field. **7**

2. (A) Prove that field has no proper ideal. **7**

2. (B) Define a subring and prove that a non-empty subset U of ring R is a sub ring if and only if following condition holds. For $a, b \in U$ (1) $a - b \in U$ and (2) $a \cdot b \in U$. **7**

OR

2. (A) State and prove the fundamental theorem of homomorphism on ring. **7**

2. (B) Define ring homomorphism. If $\Phi : (R_1, +_1, *_1) \to (R_2, +_2, *_2)$ is a ring homomorphism and I is an ideal of R_1, then prove that $\Phi(I)$ is an ideal of $\Phi(R_1)$. **7**

3. (A) Using division algorithm of $f(x)$ and $g(x) \in \mathbb{Z}_5[x]$ express $f(x)$ into the form $q(x)g(x) + r(x)$ for $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$, and $g(x) = x^2 - 2x + 3 \in \mathbb{Z}_5[x]$. **7**

3. (B) Suppose $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \in \mathbb{Z}[x]$ and suppose $\frac{p}{q}$ (gcd(p, q) = 1 is the simplest form is a solution of equation $f(x) = 0$. Then prove that $p \mid a_0$ and $q \mid a_n$. **7**

OR

3. (A) State the Eisenstein criterion for irreducibility of polynomials, and discuss the irreducibility of polynomial $8x^3 + 6x^2 - 9x + 24$ over Q. **7**

3. (B) Obtain all rational roots of the equation $2x^4 + x^3 - 10x^2 - 2x + 12 = 0$. **7**

4. (A) Define prime ideal and prove that a maximal ideal in a commutative ring with unity is also a prime ideal. **7**

4. (B) An ideal I in a commutative ring R with unity is maximal ideal iff the quotient ring R/I is a field. **7**

OR

4. (A) Prove that for given field F intersection of two subfields of F is also a subfield of F. Is union of two subfields of F a subfield of F ? **7**

4. (B) Show that $I = \langle x^3 - 3x - 1 \rangle$ is a maximal ideal in $Z_3[x]$. **7**

5. Answer in brief : (Any **seven**) **14**

(i) Give two examples of division ring.

(ii) Give two examples of one commutative and one non-commutative ring.

(iii) Give an example of division ring which is not a field.

(iv) Define a principal ideal with an example.

(v) Define kernel of homomorphism.

(vi) Give an example of subring which is right ideal but not left ideal.

(vii) Define content of a polynomial.

(viii) Is $x^2 + 1 \in R[x]$ reducible ? Justify !!

(ix) State division algorithm for polynomials.

(x) Define field, give an example of finite field.

(xi) Define prime field with an example.

(xii) Give an example of prime ideal.