Seat No. : _______________

AG-122

April-2015

M.Sc., Sem.-IV

MAT-508 : Mathematics

(Fourier Analysis)

Time : 3 Hours] **[Max. Marks : 70**

1. (A) Attempt any **one** : **7**

   (1) State and prove the uniqueness theorem for the real-valued continuous and L^1 functions.

   (2) State and prove the Riemann-Lebesgue lemma using the fact that the step functions are dense in L^1.

   (B) Attempt any **two** : **4**

   (1) Does there exist a non-constant function $f \in L^1$ such that $\hat{f}(m + n) = \hat{f}(m) + \hat{f}(n)$ for all integers m and n ?

   (2) If f is absolutely continuous then show that $\widehat{Df}(n) = in\hat{f}(n)$.

   (3) If $1 \leq p < q < \infty$, then show that $L^q \subset L^p$.

   (C) Attempt in brief : **3**

   (1) If $f(x) = ie^{ix} - 2ie^{-ix} + 1$ and $g = \bar{f}$, then what is $\hat{g}(1)$?

   (2) Show that the Fourier transform map $T : L^1 \rightarrow l_\infty(Z)$ is linear.

   (3) Give an example of a discontinuous function f such that $\hat{f}(1) = 1$ and $\hat{f}(n) = 0$ if $n \neq 1$.

2. (A) Attempt any **one** : **7**

   (1) If γ is a non-trivial complex continuous algebra homomorphisms between L^1 and the space of complex numbers C, then show that there exists a unique positive integer N such that $\gamma(f) = \hat{f}(N)$, for every $f \in L^1$.

   (2) If $f \in L^1$ and $g \in C^k$, then show that $f * g \in C^k$ and $D^j (f * g) = f * (D^j g)$, $1 \leq j \leq k$.

(B) Attempt any **two** : **4**

(1) Does there exist $f \in L^1$ such that $f * f \neq f$, but $f * f * f = f$? Justify your answer.

(2) If $f \in L^1$ and g is of bounded variation then show that $f * g$ is of bounded variation.

(3) Show that L^1 does have zero divisors with respect to convolution.

(C) Answer in brief : **3**

(1) Define the term "approximate identity".

(2) Show that convolution is distributive over addition.

(3) Give an example of an idempotent element in L^1.

3. (A) Attempt any **one** : **7**

(1) Show that

$$\|D_N\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|dx = \frac{4}{\pi^2}(\log N) + O(1).$$

(2) State and prove Fejer's theorem for continuous functions.

(B) Attempt any **two** : **4**

(1) If Σx_n is summable to 0 then show that the series is cesaro summable to 0.

(2) If $g \in L^\infty$ and $\hat{g}(n) = O(1/n)$, then show that the sequence $(\|S_N g\|_\infty)_{N=0}^{\infty}$ is a bounded sequence, where $S_N g(x) = \sum_{n=-N}^{N} \hat{g}(n) e^{inx}$.

(3) Suppose $f, g \in L^1$ and Fourier series of g converges a.e. essentially boundedly. Then show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, g(x) = \sum_{n \in Z} \hat{f}(n)\, \hat{g}(-n).$$

(C) Answer in brief : **3**

(1) If $a_n = \int_{-\pi}^{\pi} D_n(x)\, dx$ then show that (a_n) is a bounded sequence.

(2) Show that $S_N f = f * D_N$ where $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx}$.

(3) True of False : If Σc_n is Cesaro summable then it is summable (convergent).

4. (A) Attempt any **one** : **7**

(1) If $a_n \downarrow o$ then show that $\Sigma\, a_n \sin nx$ is a Fourier series of a continuous function if and only if $na_n \to 0$.

(2) If (a_n) is convex and bounded, then prove that (a_n) is decreasing and $n\Delta a_n \to 0$. Further, show that (a_n) is quasi-convex.

(B) Attempt any **two** : **4**

(1) Discuss the convergence or divergence the series $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$. Is it a Fourier series ?

(2) If $a_n \to 0$ and $\Sigma\, |\Delta a_n| < \infty$, then show that the sine series $\Sigma\, a_n \sin nx$ converges everywhere in $[-\pi, \pi]$.

(3) If $a_n \downarrow 0$ and $\Sigma \frac{a_n}{n} = \infty$, then prove that the sine series $\Sigma\, a_n \sin nx$ is not a Fourier series.

(C) Answer in brief : **3**

(1) Is $a_n = \frac{1}{n^2}$ convex ?

(2) Show that the Fourier transform map $T : L^1 \to C_0(Z)$ is not onto using the open mapping theorem.

(3) True or False : If $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (\sin nx + \cos nx)$, then f is continuous.

5. (A) Attempt any **one** : **7**

(1) State the Uniform Boundedness theorem and using it show that there exists a function which is continuous at 0 but whose Fourier series diverges at 0.

(2) State and prove Jordan's theorem.

(B) Attempt any **two** : **4**

(1) If $f \in Lip\ \alpha$, $(0 < \alpha < 1)$ then show that $S_N f(x) \to f(x)$.

(2) If (b_n) is a sequence of non-negative real numbers converging to 0, then show that there exists a sequence (a_n) of non-negative real numbers such that :

(i) $\Sigma\, a_n = \infty$, (ii) $\Sigma\, a_n b_n < \infty$ and (iii) $\Sigma \frac{a_n}{n} < \infty$

(3) State (only) some of the consequences of Jordan's theorem.

(C) Answer in brief : **3**

(1) State (only) Dini's test for convergence of Fourier series.

(2) Give an example of a function in L^∞ which cannot be factorised as $g * h$ with $g \in L^1$ and $h \in L^\infty$.

(3) Give an example of a function in L^2 which can be factorised as $g * h$ with $g \in L^2$ and $h \in L^2$.