4/81

0305N202

Candidate's Seat No :________

MSc Sem.-2 Examination

409

Mathematics

May-2025

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) If a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges when $z = z_1 (z_1 \neq z_0)$, prove that it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$. **7**

   (B) Represent the function $f(z) = \dfrac{z+1}{z-1}$ **7**

   (a) by its Maclaurin series in the domain $|z| < 1$;

   (b) by its Laurent series in the domain $1 < |z| < \infty$.

OR

   (A) Prove that a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$. **7**

   (B) Show that $\dfrac{1}{z^2 \sinh z} = \dfrac{1}{z^3} - \dfrac{1}{6} \cdot \dfrac{1}{z} + \dfrac{7}{360} z + \ldots \quad (0 < |z| < \pi)$. **7**

   Also using this expansion show that $\int_C \dfrac{dz}{z^2 \sinh z} = -\dfrac{\pi i}{3}$ when C is the positively oriented unit circle $|z| = 1$.

2. (A) Let a function f be analytic at a point z_0. Prove that it has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0, such that $f(z) = (z - z_0)^m g(z)$. **7**

   (B) Find out all singularities and determine their types and orders for the function $f(z) = \dfrac{z - \cosh z}{z^2 \cosh z}$. Also evaluate the integral $\int_{|z-i|=1} f(z)dz$. **7**

OR

   (A) State and prove Cauchy's residue theorem. **7**

   (B) Using the Cauchy's residue theorem, evaluate the integral of the function $f(z) = \dfrac{z+1}{z^2 - 2z}$ around the circle $|z| = 3$ in the positive sense. **7**

(P.T.O)

3. (A) Use residues to prove that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$. **7**

(B) Use the indented contour and $f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$ to derive the integration formula $\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a) \quad (a \geq 0,\ b \geq 0)$. **7**

OR

(A) Use residues to evaluate the integral $\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \quad (-1 < a < 1)$. **7**

(B) Use residues to prove that $\int_0^\infty \frac{\cos(ax)}{x^2+1} dx = \frac{\pi}{2} e^{-a},\ a > 0$. **7**

4. (A) Let C denote the circle $|z| = 7$, described in the positive sense. Evaluate the integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$ when the function $f(z) = \frac{(z^7+1)^5\left(z - \frac{1}{9}\right)^6}{\left(z + \frac{1}{7}\right)^{11}(z - 2i)^3}$. **7**

(B) State and prove Rouché's theorem. **7**

OR

(A) Show that the image of a circle under the transformation $w = 1/z$ maps to a circle or a straight line in the w plane and find image of circle $x^2 + y^2 = 4y$ in the w plane. **7**

(B) Find the Möbius transformation that maps the points $z_1 = -1, z_2 = i, z_3 = -i$ onto the points $w_1 = 1, w_2 = \infty, w_3 = 0$. **7**

5. **Attempt any seven of the following.** **14**

(1) The series representation of $f(z) = \frac{-1}{(z-1)(z-2)}$ in the domain $|z| < 1$ is

(A) $\sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$

(B) $\sum_{n=0}^{\infty} (2^{n-1} - 1) z^n$

(C) $\sum_{n=0}^{\infty} (2^{n+1} - 1) z^n$

(D) $\sum_{n=0}^{\infty} (2^{n+1} + 1) z^n$

(2) The Laurent series representation of $e^{1/z}$ in the domain $0 < |z| < \infty$ is

(A) $\sum_{n=0}^{\infty} \frac{1}{z^n}$

(B) $\sum_{n=0}^{\infty} \frac{1}{nz^n}$

(C) $\sum_{n=0}^{\infty} \frac{1}{n!z^n}$

(D) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

(3) The series representation of $\frac{2}{(1-z)^3}$ in the domain $|z| < 1$ is

(A) $\sum_{n=0}^{\infty} n(n+1)z^n$

(B) $\sum_{n=0}^{\infty} (n+2)z^n$

(C) $\sum_{n=0}^{\infty} (n+1)z^n$

(D) $\sum_{n=0}^{\infty} (n+1)(n+2)z^n$

(4) $\int_C e^{(z+\frac{1}{z})} dz =$ __________, where C is the circle $|z| = 1$, taken counterclockwise.

(A) $2\pi i \sum_{n=0}^{n=\infty} \frac{1}{n!}$

(B) $2\pi i \sum_{n=0}^{n=\infty} \frac{1}{n!(n+1)!}$

(C) $2\pi i \sum_{n=0}^{n=\infty} \frac{1}{(n+1)!(n+2)!}$

(D) $2\pi i \sum_{n=0}^{n=\infty} \frac{1}{n!(n+2)!}$

(5) The value of the integral of $f(z) = \frac{e^{-z}}{(z-1)^2}$ around the circle $|z|=2$ in the positive sense is

(A) $2\pi i e$

(B) $\frac{2\pi i}{e}$

(C) $-2\pi i e$

(D) $-\frac{2\pi i}{e}$

(6) At $z = 0, f(z) = \frac{1 - \cosh z}{z^3}$ has a pole of order $m =$ __________.

(A) 1

(B) 2

(C) 3

(D) 4

(7) The Jordan's inequality states

(A) $\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{2R}$ $(R > 0)$

(B) $\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{R}$ $(R > 0)$

(C) $\int_0^{\pi} e^{-R\sin\theta} d\theta > \frac{\pi}{2R}$ $(R > 0)$

(D) $\int_0^{\pi} e^{-R\sin\theta} d\theta > \frac{\pi}{R}$ $(R > 0)$

(P.T.O)

(8) To integrate $\int_0^\infty \frac{dx}{1+x^2}$, we will use a contour

(A) circle $|z| = R$

(B) real axis and unit circle $|z| = 1$

(C) real axis and upper half of circle $|z| = R$

(D) real axis and lower half of circle $|z| = R$

(9) To evaluate the integrals of the type $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$, the contour used is

(A) any circle

(B) semi-circle

(C) rectangle

(D) unit circle

(10) Let $f(z) = z^3 - 1$ and consider the contour C as the unit circle $|z| = 1$. The change in the argument of $f(z)$ as z moves once counterclockwise around C is

(A) 2π

(B) 4π

(C) 6π

(D) 0

(11) Consider the following lines $L_1 = \{z \in \mathbb{C} | \, 5x + 7y = 8\}$ and $L_2 = \{z \in \mathbb{C} | \, \text{Im}(z) = 1\}$. The images of L_1 and L_2 respectively under the inversion mapping $w = 1/z$ are

(A) Line and Circle

(B) Circles

(C) Lines

(D) Circle and Line

(12) Evaluate $\int_{|z|=3\pi} \frac{e^z}{e^z - 1} dz$

(A) πi

(B) $2\pi i$

(C) $4\pi i$

(D) $6\pi i$