Seat No. : ______________

MB-223

May-2025

M.Sc., Sem.-II

408 : Mathematics

(Real Analysis)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) If a sequence (f_n) converges in measure to f on E, and f = g almost everywhere on E, prove that (f_n) converges in measure to g. **7**

   (B) State (without proof) Luzin's theorem. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \chi_{\left[0,\ \frac{1}{3}\right]}(x) + 2\chi_{\left[\frac{1}{3},\ 1\right]}(x)$. Find a continuous function $\phi(x)$ defined on [0,1] such that $mE(f \neq \phi) < \frac{1}{4}$. (Here E = [0,1]) **7**

OR

1. (A) State (without proof) Egorov's theorem. Verify Egorov's theorem for the sequence (f_n), where $f_n : [0, 1] \to \mathbb{R}$ is defined by $f_n(x) = x^n$. **7**

   (B) Define trigonometric polynomial and its degree. Express $f(x) = (\cos^3 x + \sin x \cos x)$ as a trigonometric polynomial. **7**

2. (A) Prove that $f(x) = \frac{1}{\sqrt[3]{x}}$ belongs to $L_1[0, 8]$ but f does not belong to $L_3[0, 8]$. **7**

   (B) State and prove Minkowski's inequality in the space $L_p[a, b]$. **7**

OR

2. (A) Prove that $f(x) = \frac{1}{\sqrt[4]{x}}$ belongs to $L_1[0, 16]$ but f does not belong to $L_4[0, 8]$. **7**

   (B) Show that the set of all bounded measurable functions on [a, b] is dense in $L_p[a, b]$. **7**

3. (A) Show that a function of finite variation on [a, b] is bounded. Is the converse true ? Justify. **7**

   (B) Characterize (without proof) absolutely continuous functions on [a, b]. Is the Cantor-Lebesgue function – absolutely continuous on [0, 1] ? Justify. **7**

OR

3. (A) Prove that every absolutely continuous function on [a, b] is of finite variation. **7**

(B) Explain the construction of the Cantor-Lebesgue function Θ on [0, 1]. Is Θ absolutely continuous ? Justify your answer. **7**

4. (A) Define the Fourier series of $f \in L[-\pi, \pi]$. Give an example of $f \in L[-\pi, \pi]$ such that its Fourier coefficients $a_3 = 3$ and $b_3 = 2$ and all other a_k, b_k are zero. **7**

(B) Find the Fourier series for function f defined on $[-\pi, \pi]$ by $f(x) = |x|$. **7**

OR

4. (A) Find the Fourier series for function f defined on $[-\pi, \pi]$ by $f(x) = e^x$. **7**

(B) Does there exist $f \in L_2[-\pi, \pi]$ with Fourier coefficient $a_k = \frac{1}{\sqrt{k}}$ for all $k \geq 1$, and $b_k = 0$ for all $k \geq 1$? Justify your answer. **7**

5. Attempt any **SEVEN** of the following : **14**

(1) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}$. Then the sequence (f_n) ________.

(A) converges to 0 pointwise
(B) converges to 0 uniformly
(C) converges to 1 uniformly
(D) converges to 1 pointwise

(2) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{\sin(nx + n)}{n}$. Then _______.

(A) f_n converges to 1 uniformly
(B) f_n converges to 0 uniformly
(C) f_n converges to 1 pointwise
(D) f_n converges to 0 pointwise but not uniformly

(3) Let $E = [-1, 1]$. Let $f(x) = x^2$ on E. Then what is the measure of the set $\{x \in E : f(x) > \frac{1}{2}\}$?

(A) $2\sqrt{2}$
(B) $2 + \sqrt{2}$
(C) $2 - \sqrt{2}$
(D) None of the above

(4) What is the conjugate index of p = 7 ?

(A) $\frac{6}{7}$
(B) $\frac{5}{2}$
(C) $\frac{1}{7}$
(D) $\frac{7}{6}$

(5) Find $\|x\|_2$, where $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots..) \in l_2$.

(A) $\frac{\sqrt{\pi}}{6}$ (B) $\frac{\pi}{\sqrt{6}}$

(C) $\frac{\pi}{6}$ (D) None of these

(6) Let $f(x) = x^2$ be defined on $[-1, 1]$. What is the value of $\|f\|_3$, the norm of $f(x)$ considered as an element of $L_3[-1, 1]$?

(A) $\left(\frac{2}{7}\right)^{\frac{1}{3}}$ (B) $\frac{2}{7}$

(C) $\left(\frac{2}{7}\right)^{3}$ (D) $\left(\frac{7}{2}\right)^{\frac{1}{3}}$

(7) Let $f : [0, 2\pi] \to \mathbb{R}$ be defined by $f(x) = \sin x$. What is the value of $V_a^b(f)$?

(A) 2π (B) 2

(C) 4 (D) Infinite

(8) Which of the following is true for the sequence spaces ?

(A) $l_1 \subset l_2$ (B) $l_2 \subset l_1$

(C) $l_1 = l_2$ (D) l_1 and l_2 are not comparable.

(9) Which of the following statements are true for the Cantor-Lebesgue function Θ on $[0, 1]$?

(A) Θ is C^1 function (B) Θ is Lipschitz continuous

(C) Θ is uniformly continuous (D) Θ is absolutely continuous

(10) Which of the following sequences belong to the space l_1 ?

(A) (1,2,3,4,5,0,0,0,0,..,0,0,0,..) (B) (0,0,0,0,0,0,0,...)

(C) (1,–1,1,–1,1,–1,1,–1...) (D) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

(11) If a_k and b_k are the Fourier coefficients of $f \in L_1[-\pi, \pi]$ then ________.

(A) $a_k = b_k$, for all $k \geq 1$ (B) $a_k \geq b_k$, for all $k \geq 1$

(C) $a_k \leq b_k$, for all $k \geq 1$ (D) $a_k \to 0$ and $b_k \to 0$ as $k \to \infty$

(12) Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = \int_0^x \frac{1}{\sqrt{t}} dt$. Then _______.

(A) g is C^1 function (B) g is absolutely continuous

(C) g is not Lebesgue integrable (D) g is of finite variation.