Seat No. : _______________

BG-119

May-2015

M.Sc., Sem.-II

407 : Mathematics

(Differential Geometry – I)

Time : 3 Hours] **[Max. Marks : 70**

1. (A) For the logarithmic spiral **7**

$r(t) = (e^t \cos(t), e^t \sin(t))$,

calculate the arc-length of r starting at $r(0) = (1, 0)$.

Show that the angle between r(t) and the tangent vector at r(t) is independent of t.

OR

(i) Calculate the arc-length of the catenary $r(t) = (t, \cosh(t))$ starting at the point (0, 1).

(ii) Show that the curve $r(t) = \left(\frac{1}{3}(1-t)^{3/2}, \frac{-1}{3}(1+t)^{3/2}, \frac{t}{\sqrt{2}}\right)$ is a unit-speed curve.

(B) Answer any **two :** **4**

(i) Find a Cartesian equation of the curve $r(t) = (e^t, t^4)$.

(ii) Give a parametrization r(t) for the circle $x^2 + y^2 = 1$, such that $r(0) = (0, 1)$.

(iii) Consider the twisted cubic $r(t) = (t, t^2, t^3)$. Show that the arc-length of this curve between the points (0, 0, 0) and (1, 1, 1) is less than 4.

(C) Answer **all :** **3**

(i) Parametrize the level curve $x + y = 1$.

(ii) Parametrize the line in $\mathbb{R}^3$ given by $\begin{cases} x + y + z = 1 \\ x + y - z = 1 \end{cases}$.

(iii) Consider $r(t) = (\cos^2(t), \sin^2(t))$, $0 < t < \frac{\pi}{4}$. Is r a regular curve ?

2. (A) Compute the curvature, the torsion, the vectors t, n, b and verify the Frenet-Serret equations for the curve **7**

$$r(t) = \left(\frac{12}{13}\sin(t),\ 1 + \cos(t),\ \frac{5}{13}\sin(t)\right).$$

OR

r(t) is a regular curve with nowhere vanishing curvature such that r(0) = (0, 0, 0), r(1) = (1, 0, 0), r(2) = (0, 1, 0), r(3) = (0, 0, 1). Show that there is a point of r at which the torsion is not zero.

(B) Answer any **two :** **4**

(i) $r(t) = (\cos(t), -\sin(t))$, $-\pi < t < \pi$. Find the signed curvature of r at (1, 0).

(ii) Suppose that r(s) and δ(s) are two unit-speed curves in $\mathbb{R}^2$ with the same signed curvature at corresponding points. How are the two curves related ?

(iii) Consider $r(t) = \left(\frac{1+t^2}{t}, \frac{1-t^2}{t}, \frac{1}{t}\right)$, $t > 0$. Show that r is planar.

(C) Answer **all :** **3**

(i) Let r(t) be a regular curve in $\mathbb{R}^3$. Write down (without proof) a formula for its curvature.

(ii) Let r(t) be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Write down (without proof) a formula for its torsion.

(iii) Calculate the curvature of $r(t) = (t, t^2, t^3)$ at (0, 0, 0).

3. (A) Show that the level surface $\frac{x^2}{2^2} + \frac{y^2}{2^2} - \frac{z^2}{3^2} = 1$ is a smooth surface. **7**

OR

Show that the set $S = \{(x, y, z) \mid z = x^2 + y^3\}$ is a smooth surface with atlas consisting of the single regular surface patch $\sigma(u, v) = (u, v, u^2 + v^3)$.

(B) Answer any **two :** **4**

(i) A surface patch is given by $\sigma(u, v) = (u, v, 2u^2 - 3v^2)$. Find a basis for the tangent plane at (1, 1, –1).

(ii) Find a unit normal vector to the surface patch $\sigma(u, v) = (u, v, 2u^2 + 3v^2)$ at (0, 0, 0).

(iii) Is the vector (1, 2, 3) a tangent vector to the unit sphere at (0, 0, 1) ?

(C) Answer **all :** **3**

(i) Give a parametrization of a helicoid. (Do not prove)

(ii) Give a parametrization of the plane $2x + 3y - z = 0$. (Do not prove)

(iii) Give a parametrization of the hyperbolic paraboloid $z = x^2 - 2y^2$. (Do not prove)

4. (A) Describe the quadric $x^2 + 2y^2 + 8x - 4y + 3z = 0$. **7**

OR

Consider the curve $r(u) = (\cos(u), 0, \sin(u))$, $\frac{-\pi}{2} < u < \frac{\pi}{2}$. This curve is rotated about the z-axis. Give a parametrization for the surface of revolution thus obtained. Which surface is this ?

(B) Answer any **two :** **4**

(i) Define a generalized cylinder. Give a parametrization. (Do not prove)

(ii) Define a generalized cone. Give a parametrization. (Do not prove)

(iii) Define a ruled surface. Give a parametrization. (Do not prove)

(C) Answer **all :** **3**

(i) Define a triply orthogonal system of surfaces.

(ii) Give an example of a triply orthogonal system of surfaces, where each surface of the system is a plane.

(iii) Find the eigen values of the 3×3 matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

5. (A) Calculate the first fundamental forms of the following surfaces : **7**

(i) $\sigma(u, v) = (\sinh(u) \sinh(v), \sinh(u) \cosh(v), \sinh(u))$.

(ii) $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.

OR

Calculate the first fundamental forms of the following surfaces :

(i) $\sigma(u, v) = (\cosh(u), \sinh(u), v)$

(ii) $\sigma(u, v) = (u, v, u^2 + v^2)$

(B) Answer any **two :** **4**

(i) Is the map from the circular half-cone $x^2 + y^2 = z^2$, $z > 0$ to the xy-plane, given by $(x, y, z) \to (x, y, 0)$ an isometry ?

(ii) Give an example of a conformal map that is not an isometry.

(iii) Calculate the area of the surface (part of the unit sphere)

$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z > \frac{1}{2}\}$.

(C) Answer **all :** **3**

(i) Define an equiareal map $f : S_1 \to S_2$.

(ii) Give an isometry of the xy-plane to itself, other than the identity map.

(iii) Calculate the area of the plane region

$A = \{(x, y, z) \mid x \geq 0, y \geq 0, x + y \leq 1, z = 0\}$
