Seat No. : _______________

MA-207

May-2025

M.Sc., Sem.-II

MAT-407 : Mathematics

(Metric Spaces)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Define a metric d on a non-empty set X. Let (X, d) be a metric space and let K be a positive real number. Define $\delta(x, y) = Kd(x, y)$ for all $x, y \in X$. Show that δ is a metric on X. **7**

   (B) Define an open set in the metric space (X, d). Is the set $U = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ open in $\mathbb{R}^2$? Explain. **7**

OR

   (A) Prove that a non-empty open set in $\mathbb{R}$ is the union of countable family of pairwise disjoint open intervals. **7**

   (B) Let X be a metric space. Show that arbitrary intersections of closed sets is closed and finite union of closed sets is closed. **7**

2. (A) Let $x_k = (x_{k_1}, x_{k_2}, \ldots, x_{k_n}) \in \mathbb{R}^n$. **7**

   Show that (x_k) converges to $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ if and only if $x_{k_i} \to x_i$ as $k \to \infty$ for each i.

   (B) Let X be a metric space and let E be a subset of X. Define a limit point of E. Show that a subset E of a metric space (X, d) is closed if and only if E contains all its limit points. **7**

OR

   (A) Define Cauchy sequence in a metric space. Prove that any convergent sequence in a metric space is Cauchy. Does the converse true ? Explain. **7**

   (B) Show that the union of a finite number of bounded sets in a metric space is bounded. **7**

3. (A) Let X, Y be metric spaces. If a function $f : X \to Y$ is continuous at $x \in X$, show that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x') < \delta$, then $d(f(x), f(x')) < \varepsilon$. **7**

(B) Let X, Y be metric spaces. Show that a map $f : X \to Y$ is continuous if and only if for every open set $V \subset Y$, its inverse image $f^{-1}(V)$ is open in X. **7**

OR

(A) State and prove Urysohn's lemma. **7**

(B) Define uniformly continuous function. **7**

Discuss the uniform continuity of the following functions :

(i) $f : [1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$.

(ii) $g : (0, 1) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$.

4. (A) Show that any closed subset of a compact set in a metric space X is compact. **7**

(B) Show that $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$. **7**

OR

(A) Define : A connected metric space X. **7**

Let X be a metric space. Let A and B be two connected subsets of X such that $A \cap B \neq \phi$. Prove that $A \cup B$ is connected.

(B) Is the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ connected ? Explain. **7**

5. Attempt any **seven** of the following : **14**

(1) Let $X = \mathbb{R}$. Let $d_1(x, y) = | x^2 - y^2 |$ and $d_2(x, y) = \min 2, | x - y |$. Then

(a) d_1 is metric on $\mathbb{R}$ but d_2 is not metric on $\mathbb{R}$.

(b) d_2 is metric on $\mathbb{R}$ but d_1 is not metric on $\mathbb{R}$.

(c) d_1 and d_2 are metrics on $\mathbb{R}$.

(d) d_1 and d_2 are not metrics on $\mathbb{R}$.

(2) What is the interior of $\mathbb{Q}$ in $\mathbb{R}$?

(a) empty set

(b) $\mathbb{Q}$

(c) $\mathbb{R}$

(d) $\mathbb{R} \setminus \mathbb{Q}$

(3) Let ($\mathbb{R}$, d) be the discrete metric space. Which of the following sets are closed in ($\mathbb{R}$, d) ?

(a) $\mathbb{Q}$

(b) $\mathbb{Z}$

(c) [0, 1]

(d) (0, 1)

(4) The set of limit points of $\{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}$ is

(a) {0}

(b) empty

(c) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

(d) $\mathbb{R}$

(5) Let A, B be subsets of a metric space X. Which of the following is true ?

(a) $\overline{A \cup B} \neq \overline{A} \cup \overline{B}$

(b) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

(c) $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$

(d) If $A \subset B$, then $\overline{B} \subset \overline{A}$

(6) What is the boundary of $\mathbb{R} \times \{0\}$ in $\mathbb{R}^2$?

(a) $\mathbb{R} \times \{0\}$

(b) $\mathbb{R}^2$

(c) $\{0\} \times \mathbb{R}$

(d) Empty set

(7) Let ($\mathbb{R}$, d) be a metric space, where $d(x, y) = \frac{|x-y|}{1+|x-y|}$, then the diam($\mathbb{R}$) is

(a) 0

(b) 1

(c) 2

(d) ∞

(8) Let $A = \mathbb{N}$ and $B = \{n + \frac{1}{n} : n \in \mathbb{N}, n \geq 2\}$. Then distance $d(A, B) =$ _________

(a) 1

(b) 2

(c) ∞

(d) 0

(9) Which of the following statements is not true ?

(a) Any two open balls in $\mathbb{R}^n$ are homeomorphic.

(b) Circle and ellipse in $\mathbb{R}^2$ are homeomorphic.

(c) Circle and hyperbola in $\mathbb{R}^2$ are homeomorphic.

(d) Closed intervals [0, 1] and [0, 3] in $\mathbb{R}$ are homeomorphic.

(10) Which of the following subsets of $\mathbb{R}$ is compact ?

(a) (-1,1]

(b) $\mathbb{Z}$

(c) $\{x \in \mathbb{N} : 1 \leq x \leq 1000\}$

(d) (0, 1)

(11) Which of the following sets are compact ?

(a) The unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$

(b) The hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$

(c) $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

(d) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(12) The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is

(a) closed

(b) bounded

(c) compact

(d) connected