Seat No. : _______________

AL-133

April-2025

B.Sc., Sem.-IV

DSC-C-MAT-242 T : Mathematics (Major)

(Ring Theory)

Time : 2:00 Hours] **[Max. Marks : 50**

Instructions :
(1) **All** questions are compulsory.
(2) Write the question number in your answer book as shown in the question paper.
(3) The figure to the right indicates marks of the question.

1. (a) Prove that every finite integral domain is a field. **5**

   (b) Let U be a non-empty set with more than one element. Is $(P(U), \Delta, \cap)$ an integral domain ? Justify your answer. **5**

OR

1. (a) Define characteristic of a ring. Show that if the characteristic of an integral domain is not zero, then it must be a prime number. **5**

   (b) Let $\mathbb{Z}[i]$ denote the ring of Gaussian integers, i.e., the set of all complex numbers of the form a -t ib, where $a, b \in \mathbb{Z}$ and $i^2 = -1$. Find all units in the ring $\mathbb{Z}[i]$. **5**

2. (a) Let R be a commutative ring with unity. Prove that R is a field if it has no proper ideal. **5**

   (b) Let U be the set of all integers not divisible by 5. Is U a subring of the ring $(\mathbb{Z}, +, \bullet)$? Justify. **5**

OR

2. (a) Let R be a ring and I be an ideal of R. Prove that the map $\gamma : R \to R/I$ defined by $\gamma(a) = I + a, a \in R$ is an onto homomorphism. Determine its kernel K_ϕ. **5**

   (b) Let $I = 4\mathbb{Z}$ and $R = \mathbb{Z}$. Obtain R/I. Also, prepare the addition and multiplication table for R/I. **5**

3. (a) State and prove Eisenstein's criterion for irreducibility. **5**

   (b) Obtain the factorization of the polynomial $x^4 + 3x^3 + 2x + 3$ over $\mathbb{Z}_5$. **5**

OR

3. (a) Let F be a field and $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exist $q(x), r(x) \in F[x]$ such that $f(x) = q(x)\, g(x) + r(x)$, where either $r(x) = 0$ or $[r(x)] < [g(x)]$. Here, $[g(x)]$ means the degree of the polynomial g. **5**

(b) State factor theorem. Using it examine whether $g(x) = x + 3$ is a factor of $f(x) = x^7 + 2x + 1$ in $\mathbb{Z}_5[x]$. **5**

4. (a) Let R be a commutative ring with unity and I be an ideal of R. Prove that R/I is field iff I is maximal ideal in R. **5**

(b) Let $\mathbb{Q}[x]$ be the ring of polynomials with rational coefficients. Is $\mathbb{Q}[x] / < x^2 - 2 >$ a field ? Justify your answer with proof. **5**

OR

4. (a) Let R be a commutative ring with unity. Prove that every maximal ideal in R is a prime ideal and converse is also true if R is finite. **5**

(b) Prove that the principal ideal $I = < 4 >$ is maximal but not prime in the ring of even integers. **5**

5. Attempt any **ten** : **10**

(i) Define zero divisor in a commutative ring R.

(ii) Give an example of a ring with infinite or zero characteristic.

(iii) Give an example of a finite non-commutative ring with unity.

(iv) Define Boolean ring.

(v) State the first fundamental theorem of ring homomorphism.

(vi) Define subring of a ring.

(vii) Define content of a polynomial.

(viii) True or false ? For elements a, b if a ring R, $(a + b)^2 = a^2 + 2ab + b^2$.

(ix) Obtain an equation of degree 2 over $\mathbb{Z}_5$ that has only two solutions.

(x) True or false ? There does not exist any polynomial of degree three over the field $\mathbb{Q}$ of rational numbers which has three zeros in $\mathbb{Q}$.

(xi) What are the possible rational roots of the polynomial $2x^3 - 3x^2 + x - 6$.

(xii) Define Extension field.
