M.Sc. Semester- IV Examination
April 2025
Mathematics
MAT510EB Functional Analysis II

Time: 2.30 Hours **Marks: 70**

1. (A) Define the adjoint of an operator on H. **7**

   (B) Show that the real Banach space of all self-adjoint operators on a Hilbert space H is a partially ordered set. **7**

OR

1. (A) Define the projection on a Banach space. Give an example. **7**

   (B) Define positive operator. If A is a positive operator on a Hilbert space H, prove that $I + A$ is invertible. **7**

2. (A) State and prove Schwartz's inequality in a Hilbert space H. **7**

   (B) State (without proof) finite dimensional spectral theorem.
   If $T \in \beta(H)$ is a unitary operator on H with spectrum $\{k_1, k_2, ..k_n\}$, show that $|k_i| = 1$ for each i. **7**

OR

2. (A) Let S be a non-empty subset of a Hilbert space H, prove that $S^{\perp}$ is a closed linear subsapce of H. **7**

   (B) If P and Q are projections on M and N respectively, prove that $PQ = 0 \iff M \perp N$. **7**

3. (A) Let X be a normed linear space and $A \in BL(X)$. Define the spectrum $\sigma(A)$, eigen spectrum $\sigma_e(A)$ and the approximate eigen spectrum $\sigma_a(A)$ of A.
   Show that $\sigma_e(A) \subseteq \sigma_a(A) \subseteq \sigma(A)$. **7**

   (B) Find the spectrum of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x, 3y)$. **7**

OR

3. (A) State and prove Gelfand-Mazur theorem. **7**

   (B) Find the spectrum of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (3x, 4y)$. **7**

4. (A) Define a compact linear map. Give an example of a compact linear map. **7**

   (B) Give an example of a bounded linear operator that is not compact. **7**

OR

4. (A) If T_1 and T_2 are compact linear maps from a normed linear space X to a normed linear space Y then prove that $T_1 + T_2$ is also a compact linear map from X to Y. **7**

(B) Prove that any linear map T from $\mathbb{R}^n \to \mathbb{R}^m$ is compact. **7**

5. **Any SEVEN of the following:** **14**

(1) Let H be finite dimensional and $T \in \beta(H)$ such that $T^4 = 0$. Then

(A) $\sigma(T) = \{0\}$
(B) $\sigma(T)$ may contain a non-zero scalar.
(C) $\sigma(T)$ is empty.
(D) none of these

(2) If $P \in \beta(H)$ is a projection then which of the following statements are true?

(A) P is positive
(B) P is normal
(C) P is idempotent
(D) P is isometry

(3) Let G denote the set of all invertible matrices in algebra $M_{3\times3}(\mathbb{R})$. Then _____

(A) G is closed
(B) G is open
(C) G is connected
(D) G is disconnected

(4) Let f be defined on $[0, 2\pi]$ by $f(x) = \cos x$ then the spectrum of f, $\sigma(f)$ is _____

(A) $\mathbb{R}$
(B) $[-1, 1]$
(C) $\{0\}$
(D) $(-1, 1)$

(5) What is the dimesion of the algebra $M_{3\times3}(\mathbb{R})$, the algebra of 3×3 real matrices?

(A) 3
(B) 6
(C) 9
(D) 27

(6) Let $T \in \beta(H)$ be unitary. Then T is _____

(A) one one
(B) onto
(C) normal
(D) isometry

(7) Which one of the following is a Hilbert space?

(A) $C[0, 1]$ (B) l_2 (C) l_3 (D) $P[0, 1]$

(8) Which of the following is not a Hilbert space?

(A) l_2^n
(B) l_2
(C) $(C[0,1], ||\cdot||_2)$
(D) $L_2[0,1]$

(9) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (3x, 4y)$ is _____

(A) linear continuous
(B) linear discontinuous
(C) non-linear continuous
(D) none of these

(10) Which of the following is a Banach space?

(A) $(C[0,1], ||\cdot||_1)$
(B) $(c_0, ||\cdot||_\infty)$
(C) $(P[0,1], ||\cdot||_\infty)$
(D) $(c_0, ||\cdot||_\infty)$

(11) If x and y are non-zero vectors in a Hilbert space such that x is orthogonal to y, then _____

(A) The set $\{x, y\}$ is linearly independent
(B) The set $\{x, y\}$ is linearly dependent
(C) We get equality in Schwartz's inequality
(D) none of these

(12) In which of the following spaces, the parallelogram law holds?

(A) Linear space
(B) Normed linear space
(C) Banach space
(D) Hilbert space