6/56

1504E203

Candidate's Seat No :________

M.Sc. Sem.-4 Examination

509

Mathematics (EA)

April-2025

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Show that even perfect number is of the form $2^{k-1}(2^k - 1)$. **7**

   (B) The highest power of a prime p which divide $n!$ is $[\frac{n}{p}] + [\frac{n}{p^2}] + \cdots + [\frac{n}{p^k}]$ where $p^k \leq n < p^{k+1}$. **7**

OR

   (A) Prove that $\sum_{d|n} \phi(d) = n$, for each positive integer $n \geq 1$. Verify it for $n = 30$. **7**

   (B) Show that Fermat numbers are all relatively prime to each other. Also show that Fermat number F_5 is composite. **7**

2. (A) Prove that the integer 2^k has no primitive roots, for $k \geq 3$. **7**

   (B) Solve the following system of simultaneous congruences:
   $x \equiv 5 \pmod 6$, $x \equiv 4 \pmod{11}$, $x \equiv 3 \pmod{17}$. **7**

OR

   (A) State and prove that Chinese remainder theorem. **7**

   (B) Using a table of indices for a primitive root 11, solve the following congruences:

   (i) $7x^3 \equiv 3 \pmod{11}$

   (ii) $3x^4 \equiv 5 \pmod{11}$

   7

3. (A) If p be an odd prime and $(a, p) = 1$, then prove that the congruence $x^2 \equiv a \pmod{p^n}$, $n \geq 1$ has solution iff $(a/p) = 1$.

   7

   (B) Solve the quadratic congruence $x^2 + 7x + 10 \equiv 0 \pmod{11}$. **7**

OR

(P.T.O)

(A) State and prove Euler's criterion for deciding whether an integer a is a quadratic residue of a given prime p. 7

(B) Solve the quadratic congruence $x^2 \equiv 7 \pmod{3^3}$. 7

4. (A) Define finite simple continued fraction. Prove that any rational number can be written as a finite simple continued fraction. 7

(B) Express following rational numbers as finite simple continued fractions: 7

(i) $-\frac{19}{51}$

(ii) $\frac{71}{55}$

OR

(A) Define Pythagorean triangle. Prove that radius of the inscribed circle of a Pythagorean triangle is always an integer. 7

(B) By means of continued fractions determine the general solutions of Diophantine equation $19x + 51y = 1$. 7

5. **Attempt any seven of the following.** 14

(1) Find all incongruent solution of linear congruence $6x \equiv 15 \pmod{21}$.

(2) Find value of $\phi(n)$ and $\sigma(n)$ for $n = 5040$.

(3) Find highest power of 7 dividing 2000!.

(4) Find unit digits of 3^{100}.

(5) Find order of integers 2 and 5 under modulo 17.

(6) Find the index of 5 relative to primitive root 6 of 13.

(7) Find the value of legendre symbol $(-23/59)$.

(8) Check whether the congruence $x^2 \equiv 46 \pmod{17}$ is solvable or not.

(9) Express the rational number $\frac{71}{55}$ as finite simple continued fraction.

(10) Determine whether the quadratic congruence $x^2 + 7x + 10 \equiv 0 \pmod{11}$ is solvable or not.

(11) Determine the rational number represented by the finite simple continued fraction $[0; 2, 1, 2]$.

(12) If x, y, z is a primitive Pythagorean triple, then prove that one of the integers x or y is odd.

1504E203-3 Candidate's Seat No :________

M.Sc. Sem.-4 Examination

509

Mathematics (EB)

April-2025

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) State and prove first shifting theorem.Using first shifting theorem, obtain the Laplace transform of $5e^{2t}\sinh 2t$. **7**

   (B) Find the inverse Laplace transform of $\frac{s}{(s^2-9)^2}$. **7**

OR

   (A) Using the Laplace transform, solve the following initial value problems: **7**
   $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, \quad y(0) = 0,\ y'(0) = 0$

   (B) State second shifting theorem.(Do not prove.) Using the second shifting theorem, find the Laplace transform of $t^2u(t-1)$ and $(t-1)u(t-1)$. **7**

2. (A) Find the Fourier series of the function: **7**

$$f(x) = \begin{cases} 1 & \text{if } -\pi \le x < 0 \\ -1 & \text{if } 0 \le x \le \pi \end{cases}$$

   (B) Show that **7**

$$\int_0^\infty \frac{\cos xw + w \sin xw}{1+w^2}\,dw = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

OR

   (A) Find the Fourier cosine transform of the function $f(x) = e^{-ax}$, where $a > 0$. **7**

   (B) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem
   $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$ **7**

3. (A) Solve the difference equation using $Z-$transform $y_{n+2} - 7y_{n+1} + 10y_n = 0; n \ge 0$ with initial conditions $y(0) = 5, y(1) = 16$ **7**

(P.T.O)

(B) Find the inverse Z-transform of $\dfrac{9z^3}{(3z-1)^2(z-2)}$ by residue method **7**

OR

(A) Solve the difference equation using $Z-$transform $6y_{k+2} - y_{k-1} - y_k = 0, k \geq 0$ and $y(0) = 0, y(1) = 1$. **7**

(B) Find the inverse Z-transform of $\dfrac{3z^2+4z}{(z^2-z+1)}$, $|z| > 1$. **7**

4. (A) Prove that
$H_n\left\{\dfrac{d^2f}{dx^2}\right\} = \dfrac{s^2}{4}\left[\dfrac{n+1}{n-1}H_{n-2}(s) - 2\dfrac{n^2-3}{n^2-1}H_n(S) + \dfrac{n-1}{n+1}H_{n+2}(s)\right]$. **7**

(B) Find the Hankel transform of the function **7**

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < a, n = 0 \\ 0 & \text{if } x > a, n = 0. \end{cases}$$

OR

(A) Show that $\int_0^a x(a^2-x^2)J_0(sx)dx = \dfrac{4a}{s^3}J_1(as) - \dfrac{2a^2}{s^2}J_0(as)$. **7**

(B) Show that if $n = 0$, the Hankel transform **7**

$$H\left\{\frac{\cos ax}{x}\right\} = \begin{cases} \dfrac{1}{\sqrt{s^2-a^2}} & \text{if } s > a \\ 0 & \text{if } s < a. \end{cases}$$

5. **Attempt any seven of the following.** **14**

(1) The Laplace transform of $e^{\frac{1}{5}t}$ is

(A) $\dfrac{5}{s-5}$ (C) $\dfrac{1}{5s-1}$

(B) $\dfrac{5}{5s-1}$ (D) $\dfrac{1}{s-5}$

(2) The inverse Laplace transform of $\dfrac{1}{s^3}$ is

(A) $\dfrac{1}{t}$ (C) $\dfrac{2}{t^2}$

(B) $\dfrac{t}{2}$ (D) $\dfrac{t^2}{2}$

(3) The convolution $e^t * e^{-t}$ equals

(A) $\sinh t$

(B) $\sinh 2t$

(C) $e^t - t - 1$

(D) $e^t - t$

(4) What happens to the Fourier series of an odd function over the interval [-L, L] ?

(A) It contains only sine terms

(B) It contains only cosine terms

(C) It contains both sine and cosine terms

(D) It is zero everywhere

(5) Given the function:

$$f(x) = x^2, \quad \text{on } (-\pi, \pi),$$

which Fourier coefficients are zero?

(A) a_n

(B) b_n

(C) a_0

(D) None of the above

(6) The Fourier coefficient b_4 of the function $f(x) = x^2$, $-\pi < x < \pi$ is

(A) $\frac{\pi^2}{2}$

(B) 1

(C) 0

(D) $\frac{1}{9}$

(7) The Z–transform of ${}^nC_k$, where $0 \leq k \leq n$, is

(A) $(1+z)^n$

(B) $(1+z^{-1})^n$

(C) $(1-z)^n$

(D) $(1-z^{-1})^n$

(8) The $Z-$ transform of $\left\{\frac{a^k}{k!}\right\}$, $(k \geq 0)$, is

(A) e^{az}

(B) e^z

(C) $e^{\frac{a}{z}}$

(D) e^{az}

(9) The inverse $Z-$ transform of $\frac{2z}{(z-2)^2}$, $|z| > 2$, is

(A) $\{f(k)\} = \{k2^k\}, k \geq 0$

(B) $\{f(k)\} = \{(k+1)2^k\}, k \geq 0$

(C) $\{f(k)\} = \{2^k\}, k \geq 0$

(D) $\{f(k)\} = \{-2^k\}, k \geq 0$

(10) $H_0\left[\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right] =$ ________

(A) $-sH_0\{f(x)\}$

(B) $-sH_1\{f(x)\}$

(C) $-s^2H_0\{f(x)\}$

(D) $-s^2H_1\{f(x)\}$

(11) For $n=1$, $H(e^{-ax}) =$ ________

(A) $s(a^2+s^2)^{-3/2}$

(B) $s(a^2-s^2)^{-3/2}$

(C) $a(a^2+s^2)^{-3/2}$

(D) $a(a^2-s^2)^{-3/2}$

(12) Hankel transform of $f(x)=\frac{e^{-ax}}{x^2}$ with $xJ_1(sx)$ as kernel is

(A) $\frac{(a^2+s^2)^{1/2}-s}{s}$

(B) $\frac{(a^2+s^2)^{1/2}-s}{a}$

(C) $\frac{(a^2+s^2)^{1/2}-a}{s}$

(D) $\frac{(a^2+s^2)^{1/2}-a}{a}$
