2503N1094

Candidate's Seat No :________

M.Sc. Sem.-3 Examination

501

Mathematics

March-2025

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Prove that $C[a, b]$, the set of all real valued continuous functions defined on $[a, b]$ is a real linear space. **7**

   (B) If L is a linear space of dimension k then prove that any subset of L having $(k+1)$ elements is linearly dependent. **7**

OR

1. (A) Define linear space. Give an example (with all details) of an infinite dimensional linear space. **7**

   (B) If L is a linear space, define a linear subspace of L. Give two linear subspaces of linear space $C[a, b]$. **7**

2. (A) If $T : N \to N'$ is a linear transformation, when do we say that T is continuous? If T is continuous then prove that there exists a $k \geq 0$ such that $||T(x)|| \leq k||x||$, for all $x \in N$. **7**

   (B) Define Banach space. Give an example of a normed linear space that is not a Banach space. Give details. **7**

OR

2. (A) Define Banach space. Give an example (with all details) of a Banach space. **7**

   (B) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x + y, x - y)$. Is T linear? continuous? Justify your answer. **7**

3. (A) Define the conjugate space N^* of a normed linear space N. Show taht N^* is always complete. **7**

   (B) State and prove the closed graph theorem. **7**

OR

3. (A) Explain the natural imbedding map $x \to F_x$ from $N \to N^{**}$. Show that under this map we can regard N as a subset of N^{**}. **7**

   (B) State and prove the open mapping theorem. **7**

(P.T.O)

4. (A) State and prove Schwarz's inequality in a Hilbert space. **7**

(B) Let $T \in \beta(N)$, define its conjugate $T^* \in \beta(N^*)$.
Show that $(T_1T_2)^* = T_2^*T_1^*$ **7**

OR

4. (A) Define an orthonormal set in a Hilbert space. Prove that any orthonormal set is linearly independent. **7**

(B) Let $T \in \beta(N)$, define its conjugate $T^* \in \beta(N^*)$.
Show that $(T_1 + T_2)^* = T_1^* + T_2^*$ **7**

5. **Attempt any SEVEN of the following:** 14

(1) What is the dimension of the linear space $\mathbb{C}$ over $\mathbb{R}$?

(A) 0 (B) 1 (C) 2 (D) infinte

(2) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x+2y, 3x+4y)$. Then ________

(A) T is linear
(B) T is not linear
(C) T is invertible
(D) T is not invertible

(3) Let $L = P_n$, the linear space of all polynomials (over $\mathbb{R}$) of degree less than n. Which of the following are norms on L?

(A) $||p|| = sup\{|p'(t)|/t \in [0, 1]\}$
(B) $||p|| = \int_0^1 |p(t)|dt$
(C) $||p|| = sup\{|p(t)|/t \in [0, 1]\}$
(D) none of above

(4) Let L be 3− dimensional linear space over the field $\mathbb{Z}_2$. Then how many vectors are there in L?

(A) 5 (B) 1 (C) 9 (D) 8

(5) Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(x_1, x_2, x_3, x_4) = (x_4, x_3, x_2, x_1)$. Then ________

(A) T is linear
(B) T is continuous
(C) T is non-linear
(D) T is discontinuous

(6) A Banach space cannot have a ________ basis.

(A) finite
(B) countable infinite
(C) uncountable infinite
(D) none

(7) Which of the following subsets is/are not subspaces of $\mathbb{R}^3$?

(A) $\{(x, y, z)/x^2 + y^2 + z^2 = 0\}$
(C) $\{(x, y, z)/x + y + z = 2\}$
(B) $\{(x, y, z)/x^2 + y^2 + z^2 = 1\}$
(D) none of the above

(8) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,\ y) = (y^2,\ x^2)$ then ________

(A) T is linear but not continuous
(C) T is not linear
(B) T is linear and continuous
(D) none of the above

(9) Which of the following is a basis of linear space $\mathbb{R}^3$?

(A) $\{(1, 1, 1,), (\sqrt{2}, \sqrt{2}, \sqrt{2}), (1, 2, 3)\}$
(C) $\{(0, 0, 0), (1, e, 0), (\pi, 1, 0)\}$
(B) $\{(1, 2, 3), (4, 5, 6), (7, 8, 9), (10, 11, 12)\}$
(D) none of these

(10) Which of the following is a Hilbert space?

(A) $\mathbb{R}^n$ with $||\cdot||_1$
(C) $\mathbb{R}^n$ with $||\cdot||_2$
(B) $C[0, 1]$ with $||\cdot||_1$
(D) none of the above

(11) Which of the following is false?

(A) $(\mathbb{R}^n)^* = \mathbb{R}^n$
(C) $C[0, 1]^* = C[a, b]$
(B) $(l_2)^* = l_2$
(D) none of the above

(12) Which of the following normed linear space is not complete?

(A) $P[a, b]$ with norm $||\cdot||_2$
(C) l_1^n
(B) $\mathbb{C}^2$ with norm $||\cdot||_2$
(D) none of the above