Seat No. : _______________

FC-142

February-2025

M.Sc., Sem.-I

403 : Mathematics

(Complex Analysis-I)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Prove that $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$, if $|z_1| < 1$ and $|z_2| < 1$. **7**

(B) Find all the roots of $(8)^{\frac{1}{6}}$ in rectangular coordinates. **7**

OR

1. (A) Determine the region of Argand diagram defined by $|z - 1| + |z + 1| \leq 4$. **7**

(B) Find all zeros of the polynomial $z^4 + 4$. **7**

2. (A) Let $f(z) = u(x, y) + iv(x, y)$ and let u_x, u_y, v_x, and v_y exist at each point in some neighborhood of z_0, where $z_0 = x_0 + iy_0$. If u_x, u_y, v_x, and v_y are continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0, show that $f'(z_0)$ exists. **7**

(B) Check whether the function $u(x, y) = 2x - x^3 + 3xy^2$ is harmonic ? If so, find out a harmonic conjugate of $u(x, y)$ and hence find $f(z)$. **7**

OR

2. (A) Answer the following : **7**

(1) Derive the Cauchy-Riemann equations in polar form.

(2) For $r > 0$, $0 < \theta < 2\pi$, is $f(z) = \log z$ differentiable ? Justify your answer.

(B) Check if the following functions satisfy the reflection property : $\overline{f(z)} = f(\bar{z})$ **7**

(1) $f : |z - 2i| \leq 2 \to \mathbb{C}$, $f(z) = z^2 + 5$

(2) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z + 2 + 3i$

3. (A) Evaluate $\int_C f(z)dz$ by using the parametric representation for C, where f(z) is defined by means of the equations **7**

$$f(z) = \begin{cases} 1, & y < 0 \\ 4y, & y > 0 \end{cases}$$

and C is the arc from z = – 1 – i to z = I + i along the curve $y = x^3$.

(B) Show that $\sin^{-1} z = -i \log [iz + (1 - z^2)^{\frac{1}{2}}]$. **7**

OR

3. (A) Evaluate $\int_C f(z)dz$ by using the parametric representation for C, where $f(z) = \pi exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, 1 + i and i, the orientation of C being in the counterclockwise direction. **7**

(B) Show that $\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}$ **7**

4. (A) If a function f is entire and bounded in the complex plane, prove that f(z) is constant throughout the plane. **7**

(B) Use the extended Cauchy integral formula to evaluate $\int_C \frac{e^{2z}}{z^4} dz$, where C is the positively oriented unit circle |z| = 1. **7**

OR

4. (A) State and prove the fundamental theorem of algebra. **7**

(B) Let C denote the positively oriented boundary of square whose sides lie along Re(z) = $\pm$ 2 and Im(z) = $\pm$ 2, then evaluate the integral $\int_C \frac{z}{2z + 1} dz$. **7**

5. Attempt any **seven** of the following : **14**

(1) Which of the set of all points in the complex plane whose imaginary part is strictly negative ?

(A) Upper half plane (B) Lower half plane

(C) Right half plane (D) Left half plane

(2) $\frac{1 + 7i}{(2 - i)^2} =$ ________

(A) –2 –i (B) 1 + 2i

(C) –1 + i (D) –1 –i

(3) $|z_1 + z_2|^2 + |z_1 - z_2|^2 =$ _______

(A) $2(|z_1|^2 + 2|z_2|^2)$ (B) $2(|z_1|^2 + |z_2|^2)$

(C) $2(|z_1|^2 - 2|z_2|^2)$ (D) $2(|z_1|^2 - |z_2|^2)$

(4) At $z = 0$, the function $f(z) = \frac{1}{z}$

(A) does not satisfy Cauchy-Riemann equations.

(B) satisfies Cauchy-Riemann equations.

(C) is differentiable.

(D) None of the above

(5) If a function $f(z) = g + ih$ is analytic in domain D and its component function g satisfies the equation $g_{xx} + g_{yy} = 0$, then

(A) h is differentiable. (B) h is harmonic function.

(C) h is not harmonic function. (D) None of the above

(6) If f(z) is differentiable at $z_0 \in \mathbb{C}$, then

(A) $f'(z_0) = u_x + iv_x$ or $f'(z_0) = u_y + iv_y$

(B) $f'(z_0) = u_x + iv_y$ or $f'(z_0) = u_y + iv_x$

(C) $f'(z_0) = u_x + iv_x$ or $f'(z_0) = v_y - iu_y$

(D) $f'(z_0) = u_y - iv_x$ or $f'(z_0) = v_x + iu_y$

(7) The value of $e^{2\pm 3\pi i}$ is

(A) $-e^2$ (B) e^2

(C) e^{-2} (D) None of the above

(8) The equality $\overline{e^{iz}} = e^{i\bar{z}}$ holds for which z when $n = 0, \pm 1, \pm 2, \ldots$

(A) $z = 2n$ (B) $z = \pi + 1$

(C) $z = n\pi$ (D) None of the above

(9) The value of $e^{z+\pi i}$ is

(A) $-e^z$ (B) e^z

(C) e^{-2} (D) None of the above

(10) The value of the integral $\int_{|z-1|=1} \frac{e^z}{z^2-1} dz$

(A) 0 (B) πei

(C) $\pi i(e - e^{-1})$ (D) $-\pi ei$

(11) Which of the following statement is **FALSE** ?

(A) If a function is analytic at any point, then it's derivative of all order are analytic there too.

(B) A function that is analytic throughout a simply connected domain must have an antiderivative everywhere in the domain.

(C) The function $f(z) = z \sin z$ always possess antiderivatives.

(D) The maximum value of $|z \sin z|$ in the closed region $|z| \leq 1$ does not lie on the boundary point of the region.

(12) Which of the following domain is **NOT** a simply connected domain ?

(A) The unit circle $|z| = 1$.

(B) The annular domain between two concentric circles $1 \leq |z| \leq 2$.

(C) The whole complex plane $\mathbb{C}$.

(D) The set of points interior to a simple closed contour.
