Seat No. : _______________

FB-141

February-2025

M.Sc., Sem.-I

402 : Mathematics

(Measure and Integration)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) Define the length of an open subset G of [a, b]. If G_1 and G_2 are open subsets of [a, b], then prove that $|G_1| + |G_2| = |G_1 \cup G_2| + |G_1 \cap G_2|$. **7**

   (B) If $E_1, E_2 \subset [a, b]$, if $mE_2 = 0$, and if $E_1 \cup E_2$ is measurable, prove that E_1 is measurable. **7**

OR

1. (A) Define the length of a closed subset F of [a, b]. Is it true that if F is a closed subset of [a, b] with $| F | = 0$, then F is the empty set ? Justify your answer. **7**

   (B) If E_1 and E_2 are measurable subsets of [0, 1] and if $mE_1 = 1$, prove that $m(E_1 \cap E_2) = mE_2$. Give an example of $E_1 \subset [0, 1]$, $E_1 \neq [0, 1]$ such that $mE_1 = 1$ (Do not give proof). **7**

2. (A) If f is measurable function on [a, b], prove that kf (k is constant) is also measurable on [a, b]. **7**

   (B) Suppose f(x) is defined on [0, 1] and

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

   Prove that f is Lebesgue integrable and find $\int_0^1 f$. **7**

OR

2. (A) If (f_n) is a sequence of measurable functions on [a, b] and if $\lim_{n\to\infty} f_n(x) = f(x)$, almost everywhere on [a, b], then prove that f is measurable on [a, b]. **7**

   (B) Suppose f(x) is defined on [0, 1] and

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

   Prove or disprove that f is Riemann integrable on [0, 1]. **7**

3. (A) Let f be a bounded function on [a, b]. If f ∈ R[a, b], then prove that f ∈ L[a, b] and $R\int_a^b f = L\int_a^b f$. **7**

(B) Define f^+ (the positive part of f) and f^- (the negative part of f).

If $f : [-2, 2] \to \mathbb{R}$ is defined as $f(x) = |x| + x$, sketch the graphs of f^+ and f^-. **7**

OR

3. (A) If f is a bounded function in L[a, b] such that $f(x) \geq 0$ almost everywhere ($a \leq x \leq b$) and if $\int_a^b f = 0$ then prove that $f(x) = 0$ almost everywhere. **7**

(B) Let $f : [a, b] \to \mathbb{R}$. Define the truncated function ${}^n f$.

If $f(x) = \log \frac{1}{x}$ $(0 < x \leq 1)$, find the truncated function ${}^2 f$. **7**

4. (A) State and prove the monotone convergence theorem. **7**

(B) Show that the function

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

is Lebesgue integrable on [0, 1]. Find $\int_0^1 f$. **7**

OR

4. (A) State and prove Fatou's lemma. **7**

(B) For each $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 2n, & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

Can we interchange the limit and integral in this case ? Justify your answer. **7**

5. Attempt any **SEVEN** of the following : **14**

(1) Let E be the set of all irrational numbers in [0, 1]. Then E is ______ in [0, 1].

(a) measurable (b) bounded

(c) closed (d) open

(2) Let $f : [0, 1] \to \mathbb{R}$ be continuous. Then which of the following statements are true ?

(a) f is measurable. (b) f is bounded

(c) f is Riemann integrable (d) f is Lebesgue integrable

(3) Let $f(x) = \cos x$ be defined on the interval $[0, 2\pi]$. If $E = \{x / |f(x)| > \frac{1}{2}\}$, what is mE ?

(a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{4}$

(c) $\frac{4\pi}{3}$ (d) none

(4) Which of the following statements are true for the Cantor set ?

(a) Cantor set is countable (b) Cantor set is uncountable

(c) Cantor is closed (d) Cantor is open

(5) Let $f(x) = x, x \in [0, 1]$. Let $E = \left(\frac{1}{4}, \frac{1}{2}\right)$. Find $\int_E f$?

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$

(c) $\frac{3}{32}$ (d) none

(6) Suppose the sequence (f_n) is defined by $f_n(x) = \frac{x}{n}, x \in [0, 1]$. Which of the following statements is the most suitable to (f_n) ?

(a) $f_n \to 0$ uniformly

(b) $f_n \to 0$ pointwise but not uniformly

(c) $f_n \to 0$ pointwise a.e but not pointwise everywhere

(d) f_n does not converge to 0 a.e.

(7) If $f \in L[a, b]$ and if $g(x) = \int_a^x f(t)\,dt$, $(a \le x \le b)$, then _______

(a) g' is continuous

(b) g' = f almost everywhere on [a, b]

(c) g' = f everywhere on [a, b]

(d) none

(8) Let $f : [a, b] \to \mathbb{R}$ be bounded. D(f) denotes the set of all points in [a, b] at which f is discontinuous. Then $f \in R[a, b]$ if

(a) $mD(f) > 0$ (b) $mD(f) = 0$

(c) $D(f) = \{a, b\}$ (d) $D(f) = \{0\}$

(9) Let f be a bounded function on [a, b]. If P and Q are any two measurable partitions of [a, b] and T is a refinement of P and Q. Then which of the following statements are true ?

(a) $L(f, Q) \leq U(f, Q)$ (b) $L(f, Q) \leq L(f, T)$

(c) $U(f, P) \leq U(f, T)$ (d) $U(f, Q) \leq L(f, Q)$

(10) Which of the following is <u>false</u> about the space L[a, b] ?

(a) L[a, b] is closed under addition.

(b) L[a, b] is closed under multiplication.

(c) L[a, b] is closed under subtraction.

(d) none

(11) If f is a bounded measurable function on [a, b], then

(a) f is continuous (b) f is Riemann integrable

(c) f is Lebesgue integrable (d) none

(12) Which of the following statements are true for the function $f : [a, b] \to \mathbb{R}$:

(a) If f is continuous, then f is bounded.

(b) If f is continuous, then f is uniformly continuous.

(c) If f is continuous, then f is differentiable.

(d) If f is continuous, then f is monotone.
