

1. (A) Describe the symmetries of a square. 7  
Construct the corresponding Cayley table.

(B) Define the center of a group  $G$ . Find the center of the Dihedral group  $D_n$ . 7

OR

(A) Show that every subgroup of a cyclic group is cyclic. List all subgroups of  $\mathbb{Z}_{30}$ . 7

(B) Let  $a$  be an element of order  $n$  in a group and let  $k$  be a positive integer. Prove that  $\langle a^k \rangle = \langle a^{\gcd(n, k)} \rangle$  and  $|a^k| = \frac{n}{\gcd(n, k)}$ . 7

2. (A) Suppose that  $\phi$  is an isomorphism from a group  $G$  onto a group  $\bar{G}$ . Prove the following: 7

(i)  $G$  is cyclic if and only if  $\bar{G}$  is cyclic.

(ii) If  $\bar{K}$  is a subgroup of  $\bar{G}$ , then  $\phi^{-1}(\bar{K}) = \{g \in G \mid \phi(g) \in \bar{K}\}$  is a subgroup of  $G$ .

(B) State Lagrange's theorem. (Do not prove.) Does the converse of Lagrange's theorem hold? Justify your answer. 7

OR

(A) For every positive integer  $n$ , prove that  $\text{Aut}(\mathbb{Z}_n)$  is isomorphic to  $U(n)$ . 7

(B) Let  $G$  and  $H$  be finite cyclic groups. Prove that  $G \oplus H$  is cyclic if and only if  $|G|$  and  $|H|$  are relatively prime. 7

3. (A) Let  $G$  be a finite Abelian group and let  $p$  be a prime number that divide the order of  $G$ . Show that  $G$  has an element of order  $p$ . 7

(B) Determine all homomorphisms from  $\mathbb{Z}_{12}$  to  $\mathbb{Z}_{30}$ . 7

OR

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- (A) If a group  $G$  is the internal direct product of a finite number of subgroups  $H_1, H_2, \dots, H_n$ , prove that  $G$  is isomorphic to the external direct product of  $H_1, H_2, \dots, H_n$ . 7
- (B) Let  $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$  under multiplication modulo 96. Express  $G$  as an external and an internal direct product of cyclic groups. 7
4. (A) Let  $G$  be a nontrivial finite group whose order is a power of a prime  $p$ . Prove that  $G$  has a nontrivial center  $Z(G)$ . 7
- (B) State Sylow's third theorem. (Do not prove). Show that any group of order 255 must be cyclic. 7

OR

- (A) Prove that an integer of the form  $2 \cdot n$ , where  $n$  is an odd number greater than 1, is not the order of a simple group. 7
- (B) Discuss the simplicity of the alternating group  $A_5$ . 7
5. **Attempt any seven of the following.** 14

- (1) Which of the following groups are cyclic?
- (A) The group of integers  $\mathbb{Z}$
- (B) The group  $\mathbb{Z} \oplus \mathbb{Z}$
- (C) Dihedral group  $D_4$
- (D) The group  $U(10)$  under multiplication modulo 10.
- (2) Suppose that  $H$  is a proper subgroup of  $\mathbb{Z}$  under addition and that contains 12, 30, and 54. What are the possibilities for  $H$ ?
- (A)  $\langle 2 \rangle$                       (B)  $\langle 3 \rangle$                       (C)  $\langle 6 \rangle$                       (D)  $\langle 12 \rangle$
- (3) Consider the group  $\{5, 15, 25, 35\}$  under multiplication modulo 40. What is the multiplicative inverse of 35?
- (A) 15                      (B) 25                      (C) 35                      (D) 5

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- (4) The order of the permutation  $(1\ 2)(2\ 3\ 4)(3\ 5\ 2\ 6)$  is  
(A) 4 (B) 8 (C) 12 (D) 24
- (5) The number of automorphisms of the cyclic group of order 34 is  
(A) 1 (B) 32 (C) 17 (D) 16
- (6) Which of the following groups are isomorphic to the group  $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ ?  
(A)  $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_5$  (C)  $\mathbb{Z}_2 \oplus \mathbb{Z}_{30}$   
(B)  $\mathbb{Z}_{60}$  (D)  $\mathbb{Z}_{12} \oplus \mathbb{Z}_5$
- (7) What is the order of the factor group  $\mathbb{Z}_{60}/\langle 15 \rangle$ ?  
(A) 4 (B) 15 (C) 10 (D) 3
- (8) How many Abelian groups (up to isomorphism) are there of order 1176?  
(A) 2 (B) 4 (C) 6 (D) 8
- (9) Let  $G$  be a group of order 121, then the order of the center of  $G$  is  
(A) 1 (B) 121 (C) 11 (D) 21
- (10) In the symmetric group  $S_3$ , the conjugacy class of  $(1\ 3)$ ,  $cl((1\ 3))$  equals  
(A)  $\{(1\ 2), (1\ 3), (2\ 3)\}$   
(B)  $\{(1\ 2), (1\ 3)\}$   
(C)  $\{(1\ 2), (1\ 2\ 3), (1\ 3\ 2)\}$   
(D)  $\{(1\ 2), (2\ 3)\}$
- (11) If  $G$  is a group of order 1250, then  $G$  has a Sylow 5-subgroup of order  
(A) 5 (B) 25 (C) 125 (D) 625
- (12) Which of the following groups are simple?  
(A)  $S_4$  (B)  $\mathbb{Z}_{31}$  (C)  $D_4$  (D)  $A_5$