Seat No. : _______________

AE-124

April-2025

B.Sc., Sem.-VI

CC-309 : Mathematics

(Theory)

Time : 2:30 Hours] **[Max. Marks : 70**

Instructions : (1) **All** questions are compulsory and carry **14** marks.
(2) Figures to the right indicate marks of the question/sub question.
(3) Notations are usual.

1. (a) In a metric space, show that a subset G of X is closed if and only if its complement $X \setminus G$ is open. **7**
(b) Prove or disprove: Arbitrary intersection of open sets is open in any metric space. **7**

OR

1. (a) State and prove Cantor's intersection theorem. **7**
(b) For $r \in \{\frac{1}{2}, 1, 1.5\}$, calculate $S_r(0)$ in (X, d), where X = [0, 1] and d is usual metric. **7**

2. (a) Prove that a compact subset of a metric space is closed. **7**
(b) Let (X, d) be a discrete metric space and (Y, ρ) be any metric space. Show that every function $f : X \to Y$ is continuous. **7**

OR

2. (a) Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ be a continuous function. If $E \subset X$ is connected, then show that f(E) is connected. **7**
(b) Let $(\mathbb{R}, d)$ be a usual metric space. Give an open cover of $\mathbb{N}$ that has no finite sub-cover. Justify your answer. **7**

3. (a) Let (f_n) be a sequence of functions in R[a, b] converging uniformly to f. Prove that $f \in R[a, b]$ and **7**

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x) dx.$$

(b) Find the pointwise limit of $f_n(x) = n^2 x^n (1 - x)$; $x \in [0, 1]$ if it exists; then determine whether (f_n) converges uniformly on [0, 1]. **7**

OR

3. (a) Suppose (f_n) is a sequence of continuous functions on $E \subset \mathbb{C}$ that converges uniformly to a function f. Show that f is continuous on E. **7**

(b) Show that the series $\sum_{k=1}^{\infty} (xe^{-x})^k$ is uniformly convergent in [0, 2]. **7**

4. (a) Show that for $|x| < 1$, $\alpha \in \mathbb{R}$, **7**

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \ldots,$$

where $\binom{\alpha}{0} = 1, \binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \ldots (\alpha - k + 1)}{k!}$, $k \in \mathbb{N}$

(b) For every $x \in \mathbb{R}$ and $n \geq 0$, show that $\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1 - x)^k = nx(1 - x) \leq \frac{n}{4}$. **7**

OR

4. (a) State and prove Weierstrass approximation theorem. **7**

(b) Obtain the Taylor's expansion of $f(x) = \cos^{-1} x$ and determine its radius of convergence. **7**

5. Attempt any **seven** in short : **14**

(i) Define *dense set* in a metric space X. Is the set of all rationals dense in $\mathbb{R}$? Justify.

(ii) Show that in the usual metric space $\mathbb{R}$, the point 0 is a limit point of the set $S = \{\frac{1}{n} : n \in \mathbb{N})$.

(iii) Give an example of a metric space X and $A \subset X$ such that A has an empty interior.

(iv) Define *Cauchy sequence* in metric space.

(v) Is the union of two connected subsets of a metric space X always connected ? Explain.

(vi) If a function is continuous on a compact metric space, what can be said about its range ?

(vii) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n}$.

(viii) Give an example of a sequence of differentiable functions which converge to a function that is not differentiable.

(ix) State *identity theorem* for power series.

(x) State Taylor's theorem.

(xi) Prove that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

(xii) Define *Bernstein polynomial*.
