3/66

1204E170

Candidate's Seat No :________

M.Sc. Sem.-4 Examination

508

Mathematics

April-2025

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Let R be a commutative ring with unity and let A be an ideal of R. Show that R/A is a field if and only if A is maximal. **7**

   (B) In a commutative ring with unity, prove that every maximal ideal is prime. Does the converse true? Justify your answer. **7**

OR

1. (A) Show that the factor ring of the Gaussian integers $\mathbb{Z}[i]/\langle 2-i\rangle$ is a field. How many elements does this field have? Explain. **7**

   (B) State and prove first isomorphism theorem for rings. **7**

2. (A) State and prove Eisenstein's criterion. **7**

   (B) Determine which of the polynomials below is(are) irreducible over $\mathbb{Q}$. **7**

   (1) $x^5 + 20x^4 + 5x^3 + 10x + 35$

   (2) $(3/7)x^4 - (2/7)x^2 + (9/35)x + 3/5$

   (3) $x^5 + 5x^2 + 1$

OR

2. (A) Define unique factorization domain.
   Is $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ a unique factorization domain? Justify your answer. **7**

   (B) Define Euclidean domain.
   Prove that every Euclidean domain is a principal ideal domain. **7**

3. (A) Define splitting field of a polynomial $f(x)$ over a field F.
   Find the splitting field E of $x^4 - 2$ over $\mathbb{Q}$. Find the degree $[E : \mathbb{Q}]$. **7**

(P.T.O)

E170-2

(B) For each prime p and each positive integer n, prove that, there is, up to isomorphism, a unique finite field of order p^n. **7**

OR

3. (A) Let K be a finite extension field of the field E and let E be a finite extension field of the field F. Prove that K is a finite extension field of the field F and $[K : F] = [K : E][E : F]$. **7**

(B) Prove that if c is a constructible number, then $\sqrt{|c|}$ is also constructible. **7**

4. (A) Define fixed field of a subgroup H of the Galois group $G(E/F)$.
Let $E = (\sqrt[3]{2}, \omega)$, where $\omega^3 = 1, \omega \neq 1$. Let σ_1 be the identity automorphism of E, and let σ_2 be an automorphism of E such that $\sigma_2(\omega) = \omega^2$ and $\sigma_2(\sqrt[3]{2}) = \omega(\sqrt[3]{2})$. If $H = \{\sigma_1, \sigma_2\}$, then find the fixed field of H. **7**

(B) Define solvable group. Is dihedral group D_4 solvable? Justify your answer. **7**

OR

4. (A) Factor $x^8 - 1$ as a product of irreducible polynomials over $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$ **7**

(B) Let ω be a primitive nth root of unity. Prove that $\text{Gal}(\mathbb{Q}(\omega/\mathbb{Q})$ is isomorphic to $U(n)$. **7**

5. **Attempt any seven of the following.** **14**

(1) Which of the following rings are the integral domain?

(A) $\mathbb{Z}$ (C) $\mathbb{Z} \oplus \mathbb{Z}$

(B) $\mathbb{Z}_9$ (D) $\mathbb{Z}[i]$

(2) The number of maximal ideals of the ring $\mathbb{Z}_{120}$ is

(A) 1 (B) 2 (C) 3 (D) 4

(3) The number of ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ is

(A) 1 (C) 3

(B) 2 (D) infinite

(4) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$. What is the remainder upon dividing $f(x)$ by $g(x)$?

(A) $6x + 2$ (B) $3x + 2$ (C) $x + 2$ (D) $6x + 4$

(5) Which of the following polynomials are irreducible over $\mathbb{Z}_3$?

(A) $x^2 + 2x + 2$ (B) $x^2 + x + 1$ (C) $x^3 + x^2 + 2$ (D) $x^3 + 2x + 1$

(6) In $\mathbb{Z}[\sqrt{5}]$, $1 + \sqrt{5}$ is

(A) irreducible and prime.

(B) irreducible but not prime

(C) prime but not irreducible

(D) neither irreducible nor prime.

(7) $[\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{3}) : \mathbb{Q}] =$ ________.

(A) 3 (B) 4 (C) 6 (D) 12

(8) The minimal polynomial for $\sqrt{2} - 3\sqrt{3}$ over $\mathbb{Q}$ is

(A) $x^4 - 58x^2 + 625$ (C) $x^4 - 58x^2 + 805$

(B) $x^4 - 58x^2 + 525$ (D) $x^4 - 58x^2 + 1057$

(9) Which of the following real numbers are constructible?

(A) $\sqrt{3} + 1$ (B) $\sqrt[3]{2} + 1$ (C) π (D) $\sqrt[4]{2}$

(10) The ring $Z_3[x]/\langle p(x) \rangle$ is a field with 27 elements, where $p(x)$ is

(A) $x^4 + x + 2$ (B) $x^3 + 2x + 1$ (C) $x^3 + 1$ (D) $x^3 + x + 1$

(11) The order of the Galois group of the field $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ is

(A) 6 (B) 3 (C) 1 (D) 2

(12) Which of the following groups are solvable?

(A) The dihedral group D_6

(B) The symmetric group S_5

(C) The symmetric group S_3

(D) $(\mathbb{Z}_n, +_n)$