Seat No. : _______________

N12-122

November-2014

M.Sc., Sem.-III

501 : Mathematics

(Functional Analysis – I)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **ONE** : **7**

(1) Let M and N be subspaces of a vector space V, such that V = M + N. Show that $V = M \oplus N$ if and only if $M \cap N = \{0\}$.

(2) Let B_1 and B_2 be any two bases of a linear space V. Prove that B_1 and B_2 have the same number of elements.

(b) Attempt any **TWO** : **4**

(1) If $T : V \to V$ is non-singular linear transformation and B is a basis in V, then show that T(B) is a basis in V.

(2) Is $\{f \in C[0, 1] ; f$ is a polynomial of degree $3\}$ a subspace of C[0, 1] ? Justify !

(3) Prove or disprove : The transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 0)$ is linear.

(c) Answer in brief : **3**

(1) Give two different bases for R^3.

(2) If E is idempotent then, show that I – E is idempotent.

(3) If $T : V \to V$ is a linear transformation, is it true that T^2 is also linear ? Justify !

2. (a) Attempt any **ONE** : **7**

(1) Let M be a closed subspace of a Banach space N. Prove that the quotient space N/M is complete in quotient norm.

(2) Show that $\|x\|_\infty = \sup \{|x_n|\}$ defines a norm on 1_∞.

(b) Attempt any **TWO** : **4**

(1) Show that the norm is continuous.

(2) Draw the sets $S_i = \{x = (x_1, x_2) \in R^2 ; \|x\|_i = 1\}$ for i = 1 and 2.

(3) Prove : If T and S are in $\beta(N)$, then $\|TS\| \le \|T\| \|S\|$.

(c) Answer in brief : **3**

(1) Find the norm $\|T\|$, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x_1, x_2) = (0, x_1 + x_2)$.

(2) State Holder's inequality.

(3) Let $T : N \to N'$ be a linear transformation. Prove that if T is continuous at origin, then it is continuous everywhere.

3. (a) Attempt any **ONE** : **7**

(1) Prove : $l_p^* = l_q$ where $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(2) Prove : For $x \in$ N, the function F_x defined on N* by $F_x(f) = f(x)$, (f $\in$ N*) is in N**. Also, show that $\|F_x\| = \|x\|$.

(b) Attempt any **TWO** : **4**

(1) If N is finite dimensional then show that N* is also finite dimensional.

(2) Define separable space and give one example of it.

(3) If M is a closed subspace of N and $x \notin$ M, then show that there exists f $\in$ N* such that f(M) = 0 and $f(x) \neq 0$.

(c) Answer in brief : **3**

(1) Is it true that every non complete nls is non reflexive ?

(2) State the Hahn Banach theorem.

(3) State what is the dual space of c_0.

4. (a) Attempt any **ONE** : **7**

(1) State and prove closed graph theorem.

(2) Prove : A subset X of a nls N is bounded if and only if f(X) is a bounded set in R, for each f in N*.

(b) Attempt any **TWO** : **4**

(1) Let T be invertible β(N). Show that T* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

(2) Prove : If P is a projection on a Banach space B, then range of P and null space of P are closed in B.

(3) Prove : If T and S are in β(N), then (TS)* = S*T*.

(c) Answer in brief : **3**

(1) Define reflexive space.

(2) Show that the conjugate of an identity operator is an identity operator.

(3) Prove : If T is in β(N), then T* is linear.

5. (a) Attempt any **ONE** : **7**

(1) If M is a proper closed subspace of a Hilbert space H, then show that there exists a non-zero vector z_0 in H such that $z_0 \perp M$.

(2) Prove : A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

(b) Attempt any **TWO** : **4**

(1) Show that if M is a closed subspace of H, then $M^\perp$ is also a closed subspace of H.

(2) Prove : The Parallelogram Law in a Hilbert space.

(3) Show that the ortho-normal set $\{e_1, e_2, e_3, \ldots, e_n, \ldots\}$ in l_2 is complete.

(c) Answer in brief : **3**

(1) State Schwarz inequality.

(2) Is every inner product space, a normed linear space ? Why ?

(3) Prove : If $T \subset S$, then $S^\perp \subset T^\perp$.
