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2509N497

Candidate's Seat No :________

M.Sc Sem.-4 (Rep) Examination

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Mathematics

September-2024

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Define integral domain. Prove that a finite integral domain is a field. Is the ring $\mathbb{Z}_{24}$ an integral domain? Justify your answer. **7**

   (B) If n is an integer greater than 1, show that $\langle n \rangle = n\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ if and only if n is prime. **7**

OR

1. (A) Define a ring homomorphism from a ring R to a ring S. Determine all ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$. **7**

   (B) State and prove first isomorphism theorem for rings. **7**

2. (A) Prove that the polynomial ring $F[x]$ over a field F is a principal ideal domain. **7**

   (B) State (without proof) Eisenstein's criterion. Discuss the irreducibility of the polynomial $1 + x + x^2 + x^3 + x^4$ over $\mathbb{Q}$. **7**

OR

2. (A) Define unique factorization domain.
   Is $\mathbb{Z}[\sqrt{-6}] = \{a + b\sqrt{-6} \mid a, b \in \mathbb{Z}\}$ a unique factorization domain? Justify your answer. **7**

   (B) Define Euclidean domain.
   Prove that every euclidean domain is a principal ideal domain. **7**

3. (A) Define splitting field of a polynomial $f(x)$ over a field F. Find the splitting field E of $x^4 + x^2 + 1$ over $\mathbb{Q}$. Find the degree $[E : \mathbb{Q}]$. **7**

   (B) Let K be a finite extension field of the field E and let E be a finite extension field of the field F. Prove that K is a finite extension field of the field F and $[K : F] = [K : E][E : F]$. **7**

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OR

3. (A) If K is an algebraic extension of E and E is an algebraic extension of F, prove that K is an algebraic extension of F. **7**

   (B) Draw the subfield lattice of GF(64). **7**

4. (A) Let $F = \mathbb{Q}(\omega, \sqrt[3]{2})$. Find the Galois group $\text{Gal}(F/\mathbb{Q})$. Discuss the lattice of subgroups of $\text{Gal}(F/\mathbb{Q})$. **7**

   (B) Define solvable group.
   Give an example of a solvable group. Justify your answer. **7**

OR

4. (A) Determine the Galois group of $x^2 - 10x + 21$ over $\mathbb{Q}$. **7**

   (B) Define cyclotomic polynomial $\Phi_n(x)$. Find $\Phi_n(x)$, for $n = 1, 2, 3, 4, 5, 6$. **7**

5. **Attempt any seven of the following.** 14

   (1) Which of the following rings are integral domains?

   (A) $\mathbb{Z}$ (B) $\mathbb{Z}_{30}$ (C) $\mathbb{Z}_{19}$ (D) $\mathbb{Z} \oplus \mathbb{Z}$

   (2) Which of the following are the zero divisors of $\mathbb{Z}_{20}$?

   (A) 2 (B) 9 (C) 3 (D) 5

   (3) The number of the maximal ideals of the ring $\mathbb{Z}_{12}$ is

   (A) 1 (B) 2 (C) 3 (D) 4

   (4) The number of zeros of $x^2 + 3x + 2$ in $\mathbb{Z}_5$ is

   (A) 2 (B) 3 (C) 1 (D) 4

   (5) Which of the following polynomials are irreducible over $\mathbb{Z}_3$?

   (A) $x^2 + x$ (C) $x^3 + 2$
   (B) $x^3 + 2x + 1$ (D) $x^2 + x + 2$

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(6) The number of units of the ring of Gaussian integers $Z[i]$ is

(A) 1 (B) 2 (C) 3 (D) 4

(7) The degree of the minimal polynomial of $\sqrt{3}+\sqrt{5}$ over $\mathbb{Q}$ is

(A) 2
(B) 4
(C) 8
(D) 16

(8) $[GF(729):GF(27)] =$ ________

(A) 2 (B) 3 (C) 4 (D) 9

(9) Let p be a prime number. As a group under multiplication, the set of nonzero elements of $GF(p^n)$ is isomorphic to

(A) $\mathbb{Z}_{p^n-1}$ (B) $\mathbb{Z}_{p^n+1}$ (C) $\mathbb{Z}_{p^n}$ (D) $\mathbb{Z}_{p^{n-1}}$

(10) Which of the following statements is true?

(A) $\sqrt[3]{2}$ is constructible.
(B) $\sqrt{2}+\sqrt{3}$ is not constructible.
(C) π is constructible.
(D) $\sqrt{5}$ is constructible.

(11) The order of the Galois group of the field $\mathbb{Q}(\sqrt[4]{2}, i)$ over $\mathbb{Q}(i)$ is

(A) 16 (B) 8 (C) 4 (D) 2

(12) The Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}+\sqrt{5})/\mathbb{Q})$ is isomorphic to

(A) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (B) $\mathbb{Z}_2$ (C) $\mathbb{Z}_4$ (D) $\mathbb{Z}_2 \oplus \mathbb{Z}_4$