3/36

2103N1685

Candidate's Seat No :________

M.Sc. Semester-3 Examination

502

Mathematics

March-2024

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) State and prove one-step subgroup test.
   Let G be an Abelian group with identity e. Is $H = \{x \in G \mid x^2 = e\}$ a subgroup of G? Explain. **7**

   (B) Define center of a group G. Find the center of the Dihedral group D_n. **7**

OR

1. (A) Let a be an element of order n in a group and let k be a positive integer. Prove that $\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle$ and $|a^k| = \frac{n}{gcd(n,k)}$. **7**

   (B) Determine the subgroup lattice for the group Z_{40}. **7**

2. (A) Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Prove the following: **7**

   (i) G is Abelian if and only if $\overline{G}$ is Abelian.

   (ii) If $\overline{K}$ is a subgroup of $\overline{G}$, then $\phi^{-1}(\overline{K}) = \{g \in G \mid \phi(g) \in \overline{K}\}$ is a subgroup of G.

   (B) Determine the number of elements in A_5 of order 2, 3, and 5. **7**

OR

2. (A) Let H and K be finite subgroups of a group G. Prove that $|HK| = \frac{|H||K|}{|H \cap K|}$. **7**

   (B) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. **7**

3. (A) For any group G, prove that $G/Z(G)$ is isomorphic to $Inn(G)$. **7**

   (B) Show that the mapping ϕ from $\mathbb{C}^*$ to $\mathbb{C}^*$ given by $\phi(x) = x^4$ is a homomorphism. Find Ker (ϕ) and $\phi^{-1}(2)$. **7**

OR

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3. (A) State and prove first isomorphism theorem. **7**

   (B) Find all Abelian groups (up to isomorphism) of order 360. **7**

4. (A) Define conjugacy class of an element a in a group G. Calculate all conjugacy classes for the symmetric group S_3 and verify the class equation. **7**

   (B) Prove that a group of order 99 is Abelian. **7**

OR

4. (A) Prove that an integer of the form $2 \cdot n$, where n is an odd number greater than 1, is not the order of a simple group. **7**

   (B) Prove that there is no simple group of order 216. **7**

5. **Attempt any seven of the following.** 14

   (1) In the dihedral group D_4, if $X^2 = R_0$, then $X =$ ________.

   (A) V (B) R_{180} (C) D (D) R_{90}

   (2) Consider the group $\{5, 15, 25, 35\}$ under multiplication modulo 40. What is the multiplicative inverse of 5?

   (A) 15 (B) 25 (C) 35 (D) 5

   (3) If a cyclic group has an element of infinite order, then how many elements of finite order does it have?

   (A) 1 (C) 3

   (B) 2 (D) infinitely many

   (4) Let a belong to a group and $|a| = 100$. Then $|a^{98}|$ is

   (A) 50 (B) 25 (C) 20 (D) 10

   (5) Which of the following permutations are even?

   (A) (1 2 5) (C) (1 2 3 4)

   (B) (1 2)(1 3 4)(1 5 2) (D) (2 3 5 9 4)

N1685-3

(6) The number of automorphisms of the group $\mathbb{Z}_{20}$ is

(A) 10 (B) 15 (C) 4 (D) 8

(7) Suppose H and K are subgroups of a group G. If $|H| = 12$ and $|K| = 35$, then $|H \cap K|$ is

(A) 12 (B) 5 (C) 1 (D) 35

(8) What is the order of the factor group $\mathbb{Z}_{60}/\langle 15 \rangle$?

(A) 10 (B) 15 (C) 4 (D) 3

(9) If ϕ is homomorphism from $\mathbb{Z}_{30}$ onto a group of order 5, then $|\text{Ker } \phi|$ is

(A) 2 (B) 6 (C) 5 (D) 3

(10) In the dihedral group D_4, the conjugacy class of D, $cl(D)$ equals

(A) $\{D, V\}$

(B) $\{D, D'\}$

(C) $\{D, R_0, R_{90}, D'\}$

(D) $\{D.H\}$

(11) If G is a group of order 1250, then G has a Sylow 5−subgroup of order

(A) 5 (B) 25 (C) 125 (D) 625

(12) Which of the following groups are simple?

(A) S_4 (B) $\mathbb{Z}_{31}$ (C) $\mathbb{Z}_{19}$ (D) A_5
