2003N1663

Candidate's Seat No :________

M.Sc. Semester-3 Examination

501

Mathematics

March-2024

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Define linear space. Prove that $C[0, 1]$ is a linear space. **7**

   (B) Let a linear space L be the sum of two subspaces M and N, so that $L = M + N$. Then prove that $L = M \oplus N$ if and only if $M \cap N = \{0\}$. **7**

OR

1. (A) Define linear space. Prove that l_∞ is a linear space. **7**

   (B) Define linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$. Give two linear transformations T and S on $\mathbb{R}^2$ such that $TS = 0$ but $ST \neq 0$.

2. (A) Define normed linear space. Draw the open unit ball $B(0, 1)$ in $\mathbb{R}^2$ with respect to the norms $||\cdot||_1$ and $||\cdot||_2$. **7**

   (B) If $(N, ||\cdot||)$ is a normed linear space, show that the norm function $||\cdot|| : N \to \mathbb{R}$ is continuous. **7**

7

OR

2. (A) Define normed linear space. Show that $\mathbb{R}^n$ is a normed linear space with the norm $||\cdot||_1$. **7**

   (B) If $T : N \to N'$ is a linear continuous transformation, prove that there is $k \geq 0$ such that $||T(x)|| \leq k||x||$, for all $x \in N$. **7**

3. (A) If x is a non-zero vector in N, prove that there is a functional f in N^* such that $f(x) = ||x||$ and $||f|| = 1$. **7**

   (B) Write a short note on the natural imbedding. **7**

OR

3. (A) For a normed linear space N, what is its conjugate N^*?
   State (without proof) Hahn-Banach theorem. **7**

   (B) Consider the natural imbedding $x \to F_x$ form N to N^{**}. Show that it is linear and norm-preserving. **7**

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4. (A) Define Hilbert space. Give an example of a Hilbert space. 7

(B) For $T_1, T_2 \in \beta(N)$, Show that $(T_1T_2)^* = T_2^*T_1^*$. 7

OR

4. (A) Let $T \in \beta(N)$, define its conjugate $T^* \in \beta(N^*)$.
Show that $(T_1 + T_2)^* = T_1^* + T_2^*$. 7

(B) Let H be a Hilbert space and $S \subset H$. Define the orthogonal complement $S^\perp$ of S. Prove that $S^\perp$ is a closed linear subsapce of H. **7**

5. **Attempt any SEVEN of the following:** 14

(1) What is the dimension of the linear space $\mathbb{R}$ over $\mathbb{Q}$?

(A) 1 (B) 2 (C) 3 (D) infinite

(2) What is the dimension of the linear space $\mathbb{C}$ over $\mathbb{R}$?

(A) 1 (C) n
(B) 2 (D) infinite

(3) Which of the following is not a Banach space (in its usual norm)?

(A) l_∞ (B) $\mathbb{C}$ (C) $P[0,1]$ (D) $C[a,b]$

(4) The norm function $||\cdot|| : \mathbb{R} \to \mathbb{R}$ defined by $||x|| = |x|$ is ________

(A) bounded (C) uniformly continuous
(B) differentiable (D) discontinuous

(5) How many linear subspaces does the linear space $\mathbb{R}^2$ have?

(A) 1 (B) 2 (C) 4 (D) infinite

(6) Which of the following subsets is not a subspace of $\mathbb{R}^3$?

(A) $\{(x,y,z)/x+y+z=0\}$ (C) $\{(x,y,z)/x+y+z=2\}$
(B) $\{(x,y,z)/x+y+z=1\}$ (D) $\{(x,y,z)/x+y=1\}$

(7) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y) = (y+2, x+3)$ then ________

(A) T is not linear (C) T is linear and continuous
(B) T is linear but not continuous (D) T is a functional

(8) Which of the following is a basis of linear space $\mathbb{R}^3$?

(A) $\{(1, 1, 1,), (2, 2, 2), (3, 3, 3)\}$
(B) $\{(1, 2, 3), (4, 4, 5), (6, 7, 8), (6, 4, 2)\}$
(C) $\{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$
(D) none of these

(9) What is the dimension of $\mathbb{C}^n$ over $\mathbb{R}$?

(A) 2
(B) n
(C) $2n$
(D) $n/2$

(10) Which of the following is false?

(A) $(\mathbb{R}^n)^* = \mathbb{R}^n$
(B) $(l_2)^* = l_2$
(C) $C[0, 1]^* = C[a, b]$
(D) none

(11) Which of the following space is a Banach space?

(A) $A = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$
(B) $B = \{(x, y) \in \mathbb{R}^2 / x + y = 0\}$
(C) $C = \{(x, y) \in \mathbb{R}^2 / x + y = 1\}$
(D) none

(12) Which of the following (in its usual norm) is a Hilbert space?

(A) $\mathbb{R}^2$
(B) $P[a, b]$
(C) l_∞
(D) none of these