# 1604N200

Candidate's Seat No :________

## M.Sc. Sem. II Examination
## April-2024
## Mathematics
## MAT407 Metric Spaces

**Time: 2.30 Hours** **Max. Marks: 70**

1. (A) Define a metric $d$ on a non-empty set $X$. Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces. Does $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$ define a metric on the product set $X \times Y$? Justify your answer. **7**

   (B) Define an open ball $B(x, r)$ in the metric space $(X, d)$. Define an open set in the metric space $(X, d)$. Is the set $U = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$ open in $\mathbb{R}^2$? Explain. **7**

**OR**

1. (A) Let $(X, d)$ be any metric space. Show that there exists a bounded metric on $X$ which is equivalent to $d$. **7**

   (B) Let $X$ be a metric space. Show that arbitrary union of open sets is open. Show by means of an example that the arbitrary intersection of open sets need not be open. **7**

2. (A) Let $x_k = (x_{k1}, x_{k2}, \ldots, x_{kn}) \in \mathbb{R}^n$.
   Show that $(x_k)$ converges to $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ if and only if $x_{ki} \to x_i$ as $k \to \infty$ for each $i$. **7**

   (B) Let $X$ be a metric space and let $E$ be a subset of $X$. Define a limit point of $E$. Show that a point $x$ is a limit point of $E$ if and only if there exists a sequence $(x_n)$ in $E$ such that $x_n \to x$. **7**

**OR**

2. (A) State and prove Bolzano Weierstrass theorem. **7**

   (B) Define Cauchy sequence in a metric space. Prove that any Cauchy sequence in a metric space is bounded. **7**



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3. (A) Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces. Consider the metric space $(X \times Y, d)$, where $d$ is the product metric. Show that projections $P_X$ and $P_Y$ are continuous. **7**

(B) Let $X, Y$ be metric spaces. Show that a map $f : X \to Y$ is continuous if and only if for every open set $V \subset Y$, its inverse image $f^{-1}(V)$ is open in $X$. **7**

**OR**

3. (A) State and prove Urysohn's lemma. **7**

(B) Define homeomorphism. Show that any two closed and bounded intervals in $\mathbb{R}$ are homeomorphic. **7**

4. (A) Show that any compact subset of a metric space is closed and bounded. **7**

(B) Is the set $A = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ compact in $\mathbb{R}^2$? Justify your answer. **7**

**OR**

4. (A) Define: A connected metric space $X$.
Let $A$ be a connected subset of metric space $X$. Let $A \subset B \subset \overline{A}$. Show that $B$ is connected. **7**

(B) Let $X$ and $Y$ be connected metric spaces. Prove that the product space $X \times Y$ is connected. **7**

5. **Attempt any seven of the following.** 14

(1) Let $X = \mathbb{R}$. Let $d_1(x, y) = (x - y)^2$ and $d_2(x, y) = \sqrt{|x - y|}$. Then

(A) $d_1$ and $d_2$ are not metrics on $\mathbb{R}$.

(B) $d_1$ and $d_2$ are metrics on $\mathbb{R}$.

(C) $d_1$ is metric on $\mathbb{R}$ but $d_2$ is not metric on $\mathbb{R}$.

(D) $d_2$ is metric on $\mathbb{R}$ but $d_1$ is not metric on $\mathbb{R}$.

(2) Let $(\mathbb{R}, d)$ be the discrete metric space. Then the interior of $\mathbb{Q}$ is

(A) empty set

(B) $\mathbb{Q}$

(C) $\mathbb{R}$

(D) $\mathbb{R} \setminus \mathbb{Q}$

(3) Which of the following statements are true?

(A) Any finite subset of a metric space is open.

(B) The set of integers $\mathbb{Z}$ is closed in $\mathbb{R}$.

(C) If in a metric space we have $B(x,r) = B(y,s)$, then it always mean that $x = y$ and $r = s$.

(D) In a normed linear space $B(x,r)$ is convex.

(4) Which of the following sets are open dense subset of $\mathbb{R}$?

(A) $\mathbb{Q}$

(B) $\mathbb{R} \setminus \mathbb{Z}$

(C) $\mathbb{R} \setminus \{1,2,3,4,5\}$

(D) $\mathbb{R} \setminus \mathbb{Q}$

(5) The closure of the set $E = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Q}\}$ is

(A) $\mathbb{R}^2$

(B) $E$

(C) $\phi$

(D) $\{(x,y) \in \mathbb{R}^2 : y \in \mathbb{Q}\}$

(6) Let $B(0,1)$ be an open ball in $\mathbb{R}^n$. Let $A$ be a finite subset of $B(0,1)$. Then the boundary of the set $B(0,1) \setminus A$ is

(A) $\{x \in \mathbb{R}^n : \|x\| = 1\}$

(B) $A$

(C) $B(0,1) \setminus A$

(D) $\{x \in \mathbb{R}^n : \|x\| = 1\} \cup A$

(7) Let $(\mathbb{R}, d)$ be a metric space, where $d(x,y) = \frac{|x-y|}{1+|x-y|}$, then the diam$(\mathbb{R})$ is

(A) 0

(B) 1

(C) 2

(D) $\infty$

(8) Let $(X, d_1) = (\mathbb{R}, d_1)$, $d_1$ being the usual metric and $(Y, d_2) = (\mathbb{R}, d_2)$, $d_2$ being the discrete metric. Let $I : (\mathbb{R}, d_2) \to (\mathbb{R}, d_1)$ be the identity mapping. Then

(A) $I$ is one-to-one.

(B) $I$ is onto.

(C) $I$ is continuous.

(D) $I^{-1}$ is continuous.

(9) Parabola $y = x^2$ in $\mathbb{R}^2$ is homeomorphic to

(A) $\mathbb{R}$

(B) $(-1,1)$

(C) ellipse

(D) hyperbola

(10) Which of the following subsets of $\mathbb{R}$ is compact?

(A) $(0,1]$

(B) $\mathbb{Q}$

(C) $\{x \in \mathbb{N} : 1 \leq x \leq 100\}$

(D) $(0,1)$

(11) Let $f : X \to \mathbb{R}$ be a nonconstant continuous function on a connected metric space, then $f(X)$ is

(A) finite

(B) countably infinite

(C) singleton

(D) uncountable

(12) Which of the following subsets of $\mathbb{R}^2$ are connected?

(A) $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$

(B) $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$

(C) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(D) $\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Q}, y \notin \mathbb{Q}\}$

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