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0304N36

Candidate's Seat No :________

M.Sc Semester-4 Examination

508

Mathematics

April-2024

Time : 2-30 Hours] **[Max. Marks : 70**

1. (A) Show that the factor ring of the Gaussian integers $\mathbb{Z}[i]/\langle 2-i\rangle$ is a field. How many elements does this field have? Explain. **7**

   (B) Let R be a commutative ring with unity and let A be an ideal of R. Show that R/A is a field if and only if A is maximal. **7**

OR

1. (A) Let R be a commutative ring with unity. Suppose that the only ideals of ring R are $\{0\}$ and R. Show that R is a field. Does the converse hold? Explain. **7**

   (B) Let ϕ be a ring homomorphism from a ring R to a ring S. Prove the following: **7**

   (i) ϕ is an isomorphism if and only if ϕ is onto and Ker $\phi = \{0\}$.

   (ii) If ϕ is an isomorphism from R onto S, then ϕ^{-1} is an isomorphism from S onto R.

2. (A) Let F be a field and let $p(x) \in F[x]$. Prove that $\langle p(x)\rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F. **7**

   (B) State (without proof) mod p irreducibility test. Discuss the irreducibility of the polynomial $f(x) = x^5 + 2x + 4$ over $\mathbb{Q}$. **7**

OR

2. (A) Show that $x^2 + 1$ is irreducible over $\mathbb{Z}_3$.

   Show that every element of the field $F = \mathbb{Z}_3[x]/\langle x^2+1\rangle$ can be written in the form $ax + b + \langle x^2+1\rangle$, where $a, b \in \mathbb{Z}_3$.

   Find the inverse of $x + 1 + \langle x^2+1\rangle$ in the field F. **7**

   (B) Define Euclidean domain.

   Prove that every euclidean domain is a principal ideal domain. **7**

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3. (A) Define splitting field of a polynomial $f(x)$ over a field F. Find the splitting field E of x^4+1 over $\mathbb{Q}$. Find the degree $[E:\mathbb{Q}]$. **7**

(B) If K is an algebraic extension of E and E is an algebraic extension of F, prove that K is an algebraic extension of F. **7**

OR

3. (A) Consider $f(x) = x^3 + x^2 + 1$ over $\mathbb{Z}_2$. Let a be a zero of $f(x)$ in the field $F = \mathbb{Z}_2[x]/\langle x^3+x^2+1\rangle$. Find the other zeros of $f(x)$ in F. **7**

(B) Draw the subfield lattices of $\mathrm{GF}(3^{18})$ and $\mathrm{GF}(2^{30})$. **7**

4. (A) Let $F = \mathbb{Q}(\sqrt[4]{2}, i)$. Find the Galois group $\mathrm{Gal}(F/\mathbb{Q}(i))$. Discuss the lattice of subgroups of $\mathrm{Gal}(F/\mathbb{Q}(i))$ and the lattice of subfields of F. **7**

(B) Define solvable group. Show that S_n is solvable when $n \leq 4$. **7**

OR

4. (A) Let F be a field of characteristic 0 and let $a \in F$. If E is the splitting field of $x^n - a$ over F, prove that the Galois group $\mathrm{Gal}(E/F)$ is solvable. **7**

(B) Define cyclotomic polynomial $\Phi_n(x)$. Find $\Phi_n(x)$, for $n = 1, 2, 3, 4, 5, 6$. **7**

5. **Attempt any seven of the following.** **14**

(1) Which of the following are the zero divisors of $\mathbb{Z}_{22}$?

(A) 2 (B) 4 (C) 3 (D) 8

(2) What is the characteristic of $\mathbb{Z}[i]/\langle 2+i\rangle$?

(A) 0 (C) 5

(B) 4 (D) 2

(3) Consider the ring homomorphism ϕ from $\mathbb{C}$ onto $\mathbb{C}$ given by $a+bi \to a-bi$. Then, Ker $\phi =$ ______.

(A) $\{0\}$ (C) $\mathbb{C}$

(B) $\mathbb{R}$ (D) $\{a+bi \mid a, b \in \mathbb{Z}\}$

(4) The number of zeros of $x^2 + 3x + 2$ in $\mathbb{Z}_6$ is

(A) 2 (B) 3 (C) 1 (D) 4

(5) The ring $Z_2[x]/\langle p(x) \rangle$ is a field with 8 elements, where $p(x)$ is

(A) $x^2 + x$ (B) $x^3 + x + 1$ (C) $x^3 + 1$ (D) $x^2 + x + 1$

(6) Consider the ring of Gaussian integers $Z[i]$. Which of the following is not true?

(A) $Z[i]$ is a subring of the ring of complex numbers.

(B) $Z[i]$ is a field.

(C) $\mathbb{Z}[i]$ is a Euclidean domain.

(D) $1, -1, i$, and $-i$ are the only units of $Z[i]$.

(7) What is the splitting field of the polynomial $x^4 - x^2 - 2$ over $\mathbb{Q}$?

(A) $\mathbb{Q}(\sqrt{2})$ (B) $\mathbb{Q}(\sqrt[4]{2}, i)$ (C) $\mathbb{Q}(i)$ (D) $\mathbb{Q}(\sqrt{2}, i)$

(8) The minimal polynomial for $\sqrt{2} - \sqrt{3}$ over $\mathbb{Q}$ is

(A) $x^4 + 10x^2 + 1$ (C) $x^4 - 10x^2 + 1$

(B) $x^4 - 10x^2 - 1$ (D) $x^4 + 10x^2 - 1$

(9) $[GF(729) : GF(27)] =$ __________

(A) 2 (B) 3 (C) 4 (D) 9

(10) Which of the following real numbers are constructible?

(A) $\sqrt[3]{2}$ (B) $\sqrt{2} + \sqrt{3}$ (C) π (D) $\sqrt[4]{2}$

(11) The order of the Galois group of the field $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$ is

(A) 8 (B) 4 (C) 3 (D) 2

(12) Let $\alpha = \cos(2\pi/9) + i\sin(2\pi/9)$. Then $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is isomorphic to

(A) $\mathbb{Z}_2$ (B) $\mathbb{Z}_3$ (C) $\mathbb{Z}_6$ (D) S_3
