Seat No. : _______________

AD-131

April-2023

B.Sc., Sem.- VI

CC-309 : Mathematics

Time : 2:30 Hours] **[Max. Marks : 70**

Instruction : All questions are compulsory.

1. (A) Let X be a metric space. Prove that an open sphere is an open set. **7**

   (B) Let X be a metric space. A subset F of X is closed if and only if its complement F' is open. **7**

OR

   (A) Let (X, d) be a complete metric space and Y be a subspace of X. Then prove that Y is complete if and only if it is closed in (X, d). **7**

   (B) In a metric space X prove that every closed sphere is a closed set. **7**

2. (A) Prove that compact subsets of metric spaces are closed. **7**

   (B) Let X and Y be metric spaces and f a mapping of X into Y then prove that f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y. **7**

OR

   (A) A subset E of the real line R1 is connected if and only if it has the following property : if $x \in E$, $y \in E$ and $x < z < y$ then $z \in E$. **7**

   (B) Let (X, d_1) and (Y, d_2) be metric spaces then prove that $f : X \to Y$ is continuous on X iff $f^{-1}(F)$ is closed in X , whenever F is closed in Y. **7**

3. (A) State and prove Weierstrass M_n-test. Show that the sequence $\{f_n(x)\}$ is not uniformly convergent on any interval containing zero where $f_n(x) = \frac{nx}{1 + n^2x^2}$. **7**

   (B) Let (f_n) be a sequence of functions in R [a, b] converging uniformly to f. Then $f \in R[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$. **7**

OR

   (A) Let (f_n) be a sequence of continuous function on $E \subset C$ converges uniformly to f on E then prove that f is continuous on E. **7**

   (B) Let f_n satisfy (1) $f_n \in D[a, b]$ (2) $(f_n(x_0))$ converges for $x_0 \in D[a, b]$ (3) f_n converges uniformly on [a, b] then prove that f_n converges uniformly on [a, b] to a function f. **7**

4. (A) State and prove Abel's limit theorem. **7**

(B) For every $x \in R$, and n > 0, prove that

$$\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} = nx(1-x) \le n/4.$$ **7**

OR

(A) State and prove Weierstrass Approximation theorem. **7**

(B) State Taylor's series.

Show that for $-1 \le x \le 1$, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n} + ...$

Hence evaluate log2. **7**

5. Answer in short : (Any **SEVEN**) **14**

(1) Let X be an arbitrary non-empty set and d is defined by $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ then prove that X is a metric space.

(2) Define metric space.

(3) Define Derive set. Give one example of derive set.

(4) Define compace set.

(5) Define finite subcover.

(6) Prove that open interval (0, 1) with usual metric is not compact.

(7) Is $f_n(x) = \frac{1}{1 + nx}$ $(x \ge 0)$ continuous ? Justify.

(8) If the series Σa_k converges absolutely then prove that the series $\Sigma a_k \cos kx$ is uniformly convergent on R.

(9) Show that the series $\sum_{k=0}^{\infty}(xe^{-x})^k$ is uniformly convergent.

(10) Prove by Taylor's series $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(11) State Bernstein Theorem.

(12) For every $x \in R$ and $n \ge 0$ prove that $\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = 1$.
