Seat No. : _______________

AB-115

April-2023

B.Sc., Sem.-VI

CC-307 : Mathematics
(Abstract Algebra – II)

Time : 2:30 Hours] **[Max. Marks : 70**

Instruction : Right hand side figure indicates marks of that question.

1. (A) Define characteristic of a ring. Prove that if p is the characteristic of an integral Domain D then $(a + b)^p = a^p + b^p$; $a, b \in D$. **7**

   (B) In the set of all integers Z, the operations $\oplus$ and $\otimes$ are defined by $a \oplus b = a + b - 1$ and $a \otimes b = a + b - ab$ for all $a, b \in Z$ then show that $(Z, \oplus, \otimes)$ is a commutative ring with unity. Is an integral domain ? Is it a field ? **7**

OR

   (A) Define an integral domain and prove that every finite integral domain is a field. **7**

   (B) Show that the set $Z[i] = \{a + ib \,/\, a, b \in Z\}$ forms a ring with respect to usual addition and multiplication of complex numbers. Also show that the only elements of Z[i] have the multiplicative inverses are ± 1 , $\pm i$. Is it an Integral domain ? Is it a field ? **7**

2. (A) State and prove the fundamental theorem on homomorphism for ring. **7**

   (B) Let R be the ring of all complex number and $R' = M_{2\times2}(R) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} / a, b \in R \right\}$ be a ring w.r.t usual addition and multiplication. Define $\phi : R \to R'$ by $\phi(a + ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$; $a + ib \in R$ then Verify whether ϕ is a homomorphism or not ? Is it an Isomorphism? More over find the Kernel of ϕ. **7**

OR

   (A) Define left ideal in a ring. Prove that the Kernel of a homomorphism in ring is an ideal. **7**

   (B) Obtain all ideals of the ring $(Z_{18}, +_{18}, \bullet_{18})$. **7**

3. (A) For non-zero polynomial f, g $\in$ D[x] then in usual notation prove that [fg] = [f] + [g]. **7**

(B) Define g.c.d. of two polynomials over a field F, Using Euclid's algorithm for the polynomials $f(x) = x^3 + 2x^2 + 3x + 2$ and $g(x) = x^2 + 4$ in $Z_5[x]$, then find g.c.d. of f(x) and g(x). Also express it into the form a(x)f(x)+b(x)g(x). **7**

OR

(A) State and prove the division algorithm for polynomials. **7**

(B) Define irreducible polynomial, Also find all rational roots of an equation :

$4x^5 + x^3 + x^2 - 3x + 1 = 0$. **7**

4. (A) Define Maximal Ideal. Prove that an ideal I in a commutative ring R with unity is a maximal ideal iff the quotient ring R/I is a field. **7**

(B) For an integral domain D and the field F, the mapping $\phi : D \to F$ defined by $\phi(a) = (a, 1)$: $\forall a \in D$ where $F = \{[a, b] / (a, b) \in S, b \neq 0\}$ then show that $D \cong F$. **7**

OR

(A) Let R be a commutative ring with unity and I be an ideal of R, then prove that R/I is an integral domain iff 1 is a prime ideal. **7**

(B) Prove that the polynomial $f(x) = 3x^2 + x + 4$ is a reducible over Z_7 and also show that $f(x) = x^2 + x + 4$ is irreducible over Z_{11}. **7**

5. Answer the following in short : (ANY **SEVEN**) **14**

(1) Give an example of a division ring.

(2) Is $(z_6, +_6, \bullet_6)$ integral domain ? Justify your answer.

(3) Give an example of ring without unity but its subring with unity.

(4) Define Kernel of a homomorphism.

(5) If I = 4Z is an ideal of the ring R = (Z, +, •), then write down all the elements in quotient ring R/I. Also, solve equation (I + 2) • X = I + 3 for X $\in$ R/I.

(6) Define principal ideal.

(7) Find f+g and fg for two polynomial f = (2, 3, –5, 0, 0, 0...) $\in$ Z[x] and g = (l, 0, –2, 5, 0, 0, 0...) $\in$ Z[x].

(8) Obtain the quotient q(x) and the remainder r(x) on $f(x) = x^3 + 1$ dividing by $g(x) = x^2 + 3x - 5$ in R[x].

(9) Define a primitive polynomial.

(10) Define an extension field and give an example of it.

(11) Find a polynomial with integer co-efficient that has 1/2 and –1/3 as zeroes.

(12) Give an example of a prime ideal.