Seat No. : _______________

AC-155

April-2019

M.Sc., Sem.-II

408 : Mathematics
(Real Analysis)
(New)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) (i) State (without proof) the following theorems : **7**

(a) Lebesgue's theorem

(b) Egorov's theorem

(c) Luzin's theorem

(ii) Let $f(x) = 2 \sin(\pi x)$ be defined on [0, 1]. Find $B_3(x)$, the Bernstein polynomial of degree 3 for the function $f(x)$. **7**

OR

(i) Let $f(x) = \begin{cases} 1 , & x \in [0,1) \\ 0 , & x \in [1,2] \end{cases}$

Find a continuous function g(x) defined on E = [0, 2] such that $mE(f \neq g) < \frac{1}{3}$. **7**

(ii) Let $f_n(x) = \begin{cases} 1 , & x \in \left[0, \frac{1}{n}\right] \\ 0 , & x \in \left(\frac{1}{n}, 1\right] \end{cases}$

Show that the sequence $f_1(x)$, $f_2(x)$, $f_3(x)$ … converges in measure to the zero function, i.e., $f_n(x) \Rightarrow 0$. **7**

(B) Answer any **four** : **4**

(i) Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(ii) Write $f(x) = \sin x \cos x$ as a trigonometric polynomial.

(iii) Let $f(x) = \begin{cases} 1 \ , & x \in [-1, 0) \\ 0 \ , & x \in [0, 1] \end{cases}$

Show that there is no polynomial p(x) defined on E = [–1, 1] such that m E (f ≠ p) $< \frac{1}{4}$.

(iv) Let f(x) = | x |, x ∈ [–1 , 1]. Find a polynomial p(x) such that $| f(x) - p(x) | < \frac{1}{2}$, x ∈ [–1 , 1].

(v) Find n such that $\frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots < \frac{1}{10}$.

(vi) State the Weierstrass Theorem. (Do not prove)

2. (A) (i) Define a square summable function f(x) on E = [a, b]. Show that every square – summable function f(x) is summable, i.e., show that $L_2 \subset L_1$.

Let f(x) = x be defined on [0, 1]. Find the norm || f ||, where f is considered as an element of L_2 [0, 1]. **7**

(ii) Define the space L_2. **7**

Is the sequence $\left(1 , \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \ldots\right)$ in L_2 ?

Show that if $x \in L_2$ and $y \in L_2$, then $x + y \in L_2$.

OR

(i) State (without proof) Holder's inequality in Lp.

State (without proof) Minkowski's inequality in Lp.

State (without proof) Cauchy – Bunyakovski – Schwarz inequality in L_2. **7**

(ii) Let $f(x) = \begin{cases} 1 \ , & x \in [0, 1) \\ 0 \ , & x \in [1, 2] \end{cases}$ **7**

Show that f(x) ∈ L_2. Find a polynomial p(x) so that $\int_0^2 (f(x) - p(x))^2 \, dx < \frac{3}{2}$.

(B) Do any **four** : **4**

(i) Let p = 3. Find q, the index conjugate to p.

(ii) True or false ? The class of continuous functions is everywhere dense in L_2 [a, b]. (do not prove).

(iii) Consider $x \in L_2$, where $x = \left(1, \frac{1}{2}, \frac{1}{4}, \ldots \frac{1}{2^n}, \ldots\right)$, find $\| x \|$.

(iv) Let $x = (1, 0, 0, 0, 0, \ldots)$. Is $x \in l_p$?

(v) True or False ? The class of constant functions is everywhere dense in L_2 [a, b]. (do not prove).

(vi) Let $a_p = \int_0^1 x^p \, dx$, $p \geq 1$.

Find $\lim\limits_{p \to \infty} \sqrt[p]{a_p}$

3. (A) (i) Suppose f(x) is an increasing function defined on [a, b]. State (without proof) a theorem describing the set of points of discontinuity of f(x).

Let $f(x) = \begin{cases} x, & x \in [0, 1) \\ x+1, & x \in [1, 2] \end{cases}$

Sketch (roughly) the graph of f(x). Find s(x), the Saltus function of f(x). **7**

(ii) Suppose f(x) is an increasing function defined on [a, b]. State (without proof) a theorem describing the set of points at which f(x) has a finite derivative $f'(x)$.

Let $f(x) = \begin{cases} x, & x \in [0, 1) \\ x+1, & x \in [1, 2] \end{cases}$

Find the set of points at which f(x) has a finite derivative $f'(x)$. **7**

OR

(i) Define a function $f(x)$ of finite variation on [a, b]. Show that a function $f(x)$ defined on [a, b] which satisfies a Lipschitz condition is of finite variation. **7**

Shaw that the sum of two functions of finite variation is also of finite variation.

(ii) Define an absolutely continuous function $f(x)$ on [a, b]. Show that if a function $f(x)$ defined on [a, b] satisfies a Lipschitz condition, then $f(x)$ is absolutely continuous. **7**

Show that the sum of two absolutely continuous functions is absolutely continuous.

(B) Do any **three** : **3**

(i) True or false ? Every absolutely continuous function has finite variation. (do not prove)

(ii) Let $f(x) = |x|$, $x \in [-1, 1]$. Write $f(x)$ as a difference of two increasing functions.

(iii) True or false ? Every continuous function is absolutely continuous. (do not prove)

(iv) True or false ? Every absolutely continuous function is continuous. (do not prove)

(v) True or false ? If the derivative of an absolutely continuous function $f(x)$ is zero almost everywhere, then the function $f(x)$ is constant. (do not prove).

4. (A) (i) State and prove the Riemann – Lebesgue theorem. **7**

(ii) State (without proof) a necessary and sufficient condition for a function $\phi(x)$ to be the indefinite integral of a summable function. **7**

Show that every point of continuity of a summable function f(t) is a Lebesgue point of f(t).

OR

(i) Find the Fourier series for

$$f(t) = \frac{t^2}{4} \quad , \quad (-\pi \leq t \leq \pi).$$

Find the sum $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ **7**

(ii) Find the sum $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$ **7**

(B) Do any **three** : **3**

(i) Define an even function on $[-\pi, \pi]$.

(ii) Find b_τ (the Fourier coefficient) of

$f(x) = x^2$ defined on $[-\pi, \pi]$. (do not prove)

(iii) Let $f(x) = \begin{cases} 1 & , \quad x \in [-\pi, 1) \\ 0 & , \quad x = 0 \\ -1 & , \quad x \in (0, \pi] \end{cases}$

Find $f_r'(0)$.

(iv) Find $\int_{-\pi}^{\pi} \cos(2x) \sin(3x)\, dx$.

(v) Sketch (roughly) the function $f(x) = \sin 2x, -\pi \leq x \leq \pi$.