Seat No. : ______________

MM-120

March-2019

B.Sc. Sem.-VI

307 : Mathematics

(Abstract Algebra-II)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (A) (1) Define an Integral Domain. Prove that every finite integral domain is a field. **7**

(2) Let Q be ring of real quaternion's and let $a = 2 + 3i - 5j + 8k$: $b = 2 + 2i + 5j - 2k$ and $c = i + j$ are elements in Q then obtain :

(i) $a + b + c$

(ii) bc

(iii) $|b|$

(iv) multiplicative inverse of a **7**

OR

(1) Define an unit element in ring R. In usual notations prove that if R is a ring with unity then : **7**

(i) $a0 = 0a = 0$ for every $a \in R$

(ii) $(-1)(-1) = 1$

(2) Prove that the characteristic of a ring R with unity is n if and only if n is the smallest positive integer with $n1 = 0$. **7**

(B) Attempt any **two** : **4**

(1) If R is a ring with $a^2 = a$ for each $a \in R$ then show that R is commutative.

(2) Give an example of ring elements a and b with the properties that $ab = 0$, but $ba \neq 0$.

(3) Let $Z_3[i] = \{a + ib/a, b \in Z_3\} = \{0, 1, 2, i, 1 + i, 2 + i, 2i, 1 + 2i, 2 + 2i\}$ where $i^2 = -1$ be the ring of Gaussian integers modulo 3. Find the multiplicative inverse for $a = 1 + 2i$ in $Z_3[i]$.

2. (A) (1) Define homomorphism between two rings. Suppose $\phi : (R, +, \cdot) \to (R, \oplus, \odot)$ be homomorphism and if I is an ideal of R then prove that $\phi(I)$ is an ideal of $\phi(R)$. **7**

(2) Give an example of a left ideal but not a right ideal in a ring R. **7**

OR

(1) Prove that a non-empty subset I of a ring R is an ideal of R if and only if the following two conditions hold : **7**

(i) $a - b \in R$ for $a, b \in I$

(ii) ar and ra $\in I$

for $a \in I$ and $r \in R$

(2) Obtain all ideals of ring $(Z_{15}, +_{15}, \cdot_{15})$ and prepare tables for the coresponding quotient rings. **7**

(B) Attempt any **two** : **4**

(1) Let $R = (C, +, \cdot)$ and $R' = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} / a, b \in R \right\}$ are two rings and if a mapping $\phi : R \to R'$ as $\phi(a + ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for every $a + ib \in R$ then show that ϕ is a homomorphism.

(2) Define Kernel of a homomorphism.

(3) If I = 4Z is an ideal of the ring $R = (Z, +, \cdot)$ then write down all the elements in quotient ring R / I. Also, solve equation $(I + 2) \cdot X = I + 3$ for $X \in R / I$.

3. (A) (1) Define degree for a polynomial $f(x)$ in $D[x]$.

In usual notation, prove that $[fg] = [f] + [g]$ for $f, g \in D[x]$. **7**

(2) Find the g.c.d. of $f(x) = x^5 + 3x^3 + x^2 + 2x + 2$ and $g(x) = x^4 + 3x^3 + 2x^2 + x + 2$ in $Z_5 [x]$. Also, express it in the form $a(x)\, f(x) + b(x)\, g(x)$. **6**

OR

(1) State and prove division algorithm theorem for polynomials. **7**

(2) Obtain all rational zeroes of the polynomial $f(x) = 2x^3 + 22x^2 - 23x + 12$. **6**

(B) Attempt any **two** : **4**

(1) Verify irreducibility for the polynomial $f(x) = x^2 + 6$ over the field Z_5 and Z_7.

(2) Suppose f = (1, 1, 2, 3, 0, 0,...) and g = (2, 0, –3, 0, 4, 0, 0,...) $\in Z[x]$ then find $f + g$ and $f \cdot g$.

(3) Obtain the quotient $q(x)$ and remainder $r(x)$ on dividing $f(x) = 3x^3 + 2x^2 + x + 1$ by $g(x) = x^2 + 3x + 2$ in $Z_5[x]$.

4. (A) (1) Prove that an ideal I in a commutative ring R with unity is a maximal ideal if and only if the quotient ring R/I is a field. **7**

(2) Find all maximal and prime ideals in $(Z_{36}, +_{36}, \bullet_{36})$. **6**

OR

(1) Prove that an ideal I in a commutative ring R with unity is a prime ideal if and only if the quotient ring R/I is an integral domain. **7**

(2) Give an example of a prime ideal which is not a maximal ideal in ring. **6**

(B) Attempt any **two** : **4**

(1) Give an example of a finite field containing eight elements.

(2) Prove that if F is a finite field with p^n elements then the mapping $\phi : F \to F$ defined by $\phi(x) = x^p$; $x \in F$ is an automorphism of order n.

(3) Prove that the ideal $I = \langle x^3 - x - 1 \rangle$ is a maximal ideal in $Z_3[x]$.
