Seat No. : _______________

SI-132

September-2020

B.Sc., Sem.-VI

CC-307 : Mathematics
(Abstract Algebra-II)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (i) Attempt any **three** questions in Section-I.

(ii) Section-II is a compulsory section of short questions.

(iii) Notations are usual everywhere.

(iv) The right hand side figures indicate marks of the sub-question.

SECTION – I

Attempt any **THREE** of the following questions :

1. (a) Define a ring. Also prove the following properties in a ring R :

(1) $a \cdot 0 = 0 \cdot a = 0, \forall\, a \in R$, where 0 is the zero element of R.

(2) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b), \ \forall a, b \in R$. **7**

(b) Show that the set $Z(\sqrt{2}) = \{a + b\sqrt{2} / a, b \in Z\}$ forms a ring under usual addition and multiplication of real numbers. **7**

2. (a) Prove that every field is an integral domain.

Also give an example of an integral domain which is not a field. **7**

(b) Define a Boolean ring and prove that a Boolean ring is a commutative ring.

Also give an example of a Boolean ring. **7**

3. (a) Define an ideal of a ring R. Also prove that a nonempty subset I of a ring R is an ideal of R if and only if (i) $a - b \in I$, for all a, $b \in I$ and (ii) $a \cdot r$ and $r \cdot a \in I$, for all $a \in I$ and for all $r \in R$. **7**

(b) Show that (Z, +, •), the ring of integers is a principal ideal ring. **7**

4. (a) Prove that a field has no proper ideal. **7**

(b) Define a ring Homomorphism. If $\Phi : (R, +, \bullet) \to (R', \oplus, \odot)$ is a ring homomorphism and I is an ideal of R then prove that $\Phi(I)$ is an ideal of $\Phi(R')$. **7**

5. (a) For nonzero polynomials f, $g \in D[x]$ prove that [fg] = [f] + [g]. **7**

(b) Using Division algorithm for $f(x)$ and $g(x) \in Z_5[x]$ express $f(x)$ into the form $q(x)\,g(x) + r(x)$ for $f(x) = x^4 + 3x^2 + 2x + 4$ and $g(x) = x + 1 \in Z_5[x]$. **7**

6. (a) Suppose $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \in Z[x]$ and suppose $\frac{p}{q}$ in the simplest form (i. e. (p, q) = 1) is a solution of the equation $f(x) = 0$. Then prove that $p|a_0$ and $q|a_n$. **7**

(b) Show that the polynomial $x^3 + 3x^2 - 8$ is irreducible over Q. **7**

7. (a) If $\oplus$ and $\odot$ are binary operations defined on the set R of all real numbers as $a \oplus b = a + b - 1$; $a \odot b = a + b - ab$, then show that $(R, \oplus, \odot)$ is a field. **7**

(b) If F_1 and F_2 are subfields of a field F, then prove that $F_1 \cap F_2$ also is a subfield of F. **7**

8. (a) If M is a maximal ideal of a commutative ring R with unity then prove that the quotient ring R/M is a field. **7**

(b) If I = < 4 > then show that I is a maximal but not a prime ideal of the ring 2Z of all even integers. **7**

SECTION – II

9. Attempt any **FOUR** of the following in short : **8**

(i) Give an example of a division ring which is not a field.

(ii) Give an example of a subring which is not an ideal.

(iii) Give an example of a subring of a ring which is not an ideal of the ring.

(iv) Give an example of a division ring which is not a field.

(v) State the remainder theorem and the factor theorem for polynomials.

(vi) Define a prime ideal and give an example of a prime ideal.