Seat No. : _______________

JB-107

July-2021

B.Sc., Sem.-VI

307 : Mathematics
(Abstract Algebra – II)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (i) Attempt any **THREE** questions in **Section – I**.

(ii) **Section – II** is a **compulsory** section of short questions.

(iii) Notations are usual everywhere.

(iv) The right hand side figures indicate marks of the sub question.

SECTION – I

Attempt any **THREE** of the following questions :

1. (a) Define a ring with unity. If R is a ring with unity 1 then prove the followings properties in R :

(1) $a\cdot(-b) = (-a)\cdot b = -(a \cdot b), \quad \forall a, b \in R$.

(2) $(-1) \cdot (-1) = 1$. **7**

(b) Show that the set $A = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} / \alpha \in Z \right\}$ forms an integral domain under usual addition and multiplication of matrices. **7**

2. (a) Prove that every finite integral domain is a field. Also give an example of an infinite integral domain which is not a field. **7**

(b) Define a Boolean ring and prove that a Boolean ring is a commutative ring. Also give an example of a Boolean ring. **7**

3. (a) Define a subring and prove that a non empty subset U of a ring R is a subring of R if and only if (i) $a - b \in U$ and (ii) $a \cdot b \in U$ for all $a, b \in U$. **7**

(b) Show that $U = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} / \alpha \in R \right\}$ is a subring $M_2(R)$. Is it a field ? Justify your answer. **7**

4. (a) Prove that a field has no proper ideal. **7**

(b) Define a ring Homomorphism. If $\Phi : (R, +, \bullet) \rightarrow (R', \oplus, \odot)$ is a ring homomorphism and I is an ideal of R then prove that $\Phi(I)$ is an ideal of $\Phi(R')$. **7**

5. (a) For nonzero polynomials $f, g \in D[x]$ Prove that $[f \cdot g] = [f] + [g]$. **7**

(b) Using Division algorithm of $f(x)$ and $g(x) \in Z_5[x]$ express $f(x)$ into the form $q(x)$ $g(x) + r(x)$ for $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $g(x) = x^2 - 2x + 3 \in Z_5[x]$. **7**

6. (a) Suppose $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \in Z[x]$ and suppose $\frac{p}{q}$ in the simplest form (i. e. $(p, q) = 1$) is a solution of the equation $f(x) = 0$. Then prove that $p|a_0$ and $q|a_n$. **7**

(b) State the Eisenstein's criterion and prove that whenever p is a prime then $f(x) = x^{p-1} + x^{p-2} + ... + x + 1$ is irreducible over Q. **7**

7. (a) Show that $Q[i] = \{a + bi / a, b \in Q\}$ is a subfield of the field C of complex numbers. **7**

(b) If F_1 and F_2 are subfields of a field F then prove that $F_1 \cap F_2$ also is a subfield of F. **7**

8. (a) Define a prime ideal.

Also prove that a maximal ideal in a commutative ring with unity is also a prime ideal. **7**

(b) Show that $I = < x^3 - 3x - 1 >$ is a maximal ideal in $Z_3[x]$. **7**

SECTION – II

9. Attempt any **FOUR** of the fallowing in short : **8**

(i) Give an example of a division ring which is not a field.

(ii) Give an example of a finite non-commutative ring and an infinite commutative ring.

(iii) Give an example of a subring of a ring which is not an ideal of the ring.

(iv) Give an example of a division ring which is not a field.

(v) Define a polynomial in integral domain D and the degree of a nonzero polynomial in D.

(vi) Define a Principal Ideal and give one example of it.
